

# Calculating Yaw of Repose and Spin Drift

-A novel and practical approach for computing the Spin Drift perturbation-

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## Introduction

The Yaw of Repose angle  $\beta_R$  is a very small, but gradually increasing, horizontally rightward, aircraft-type yaw-attitude bias or “side-slip” angle of the *coning axis* of a right-hand spinning bullet. The Yaw of Repose reverses sign and angles leftward for a left-hand spinning bullet.

We discuss only right-hand spinning bullets here for clarity. It can be shown that for right-hand twist, the yaw of repose lies to the right of the trajectory. Thus the bullet cones around with an average attitude offset to the right, leading to increasing side drift to the right.

For spin-stabilized bullets, this attitude angle creates the well known Spin Drift. The small horizontally rightward Yaw of Repose angle causes a small rightward aerodynamic lift force which, in turn, causes a slowly increasing horizontal velocity of the bullet.

It is important to realize that this effect occurs independently of the presence of surface wind of any force or from any direction. Bear in mind that the Yaw of Repose represents

the horizontal yaw attitude of the bullet's coning axis or the average yaw of the coning bullet.

The acceleration of gravity acting upon the flat-fired bullet in free flight is the original cause of this small yaw-attitude bias angle. The downward curving of the trajectory due to gravity causes the airstream passing over the bullet to approach from below the nose of that bullet.

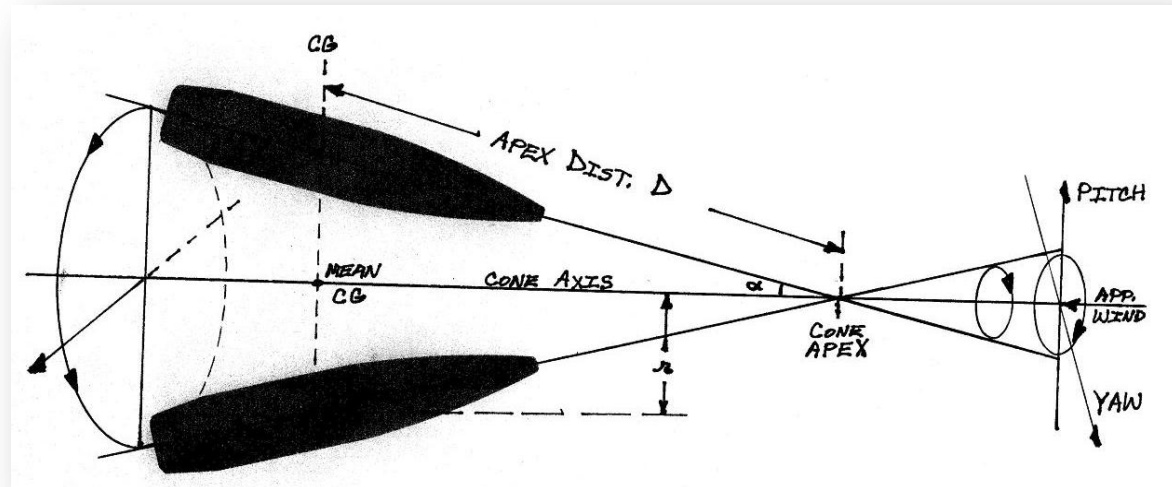


FIGURE: Extreme TDC and BDC Positions of Coning Bullet

This wind shift during each coning cycle causes an increased aerodynamic angle of attack which peaks when the center of gravity (CG) of the bullet is at the Bottom Dead Center (BDC) position in each coning cycle where its nose is oriented maximally upward.

Since the coning angle always exceeds this small change in the approaching airstream direction during each coning cycle, the aerodynamic angle of attack for a bullet at Top Dead Center (TDC) when the bullet's nose is pointing maximally downward is at a minimum for that coning cycle.

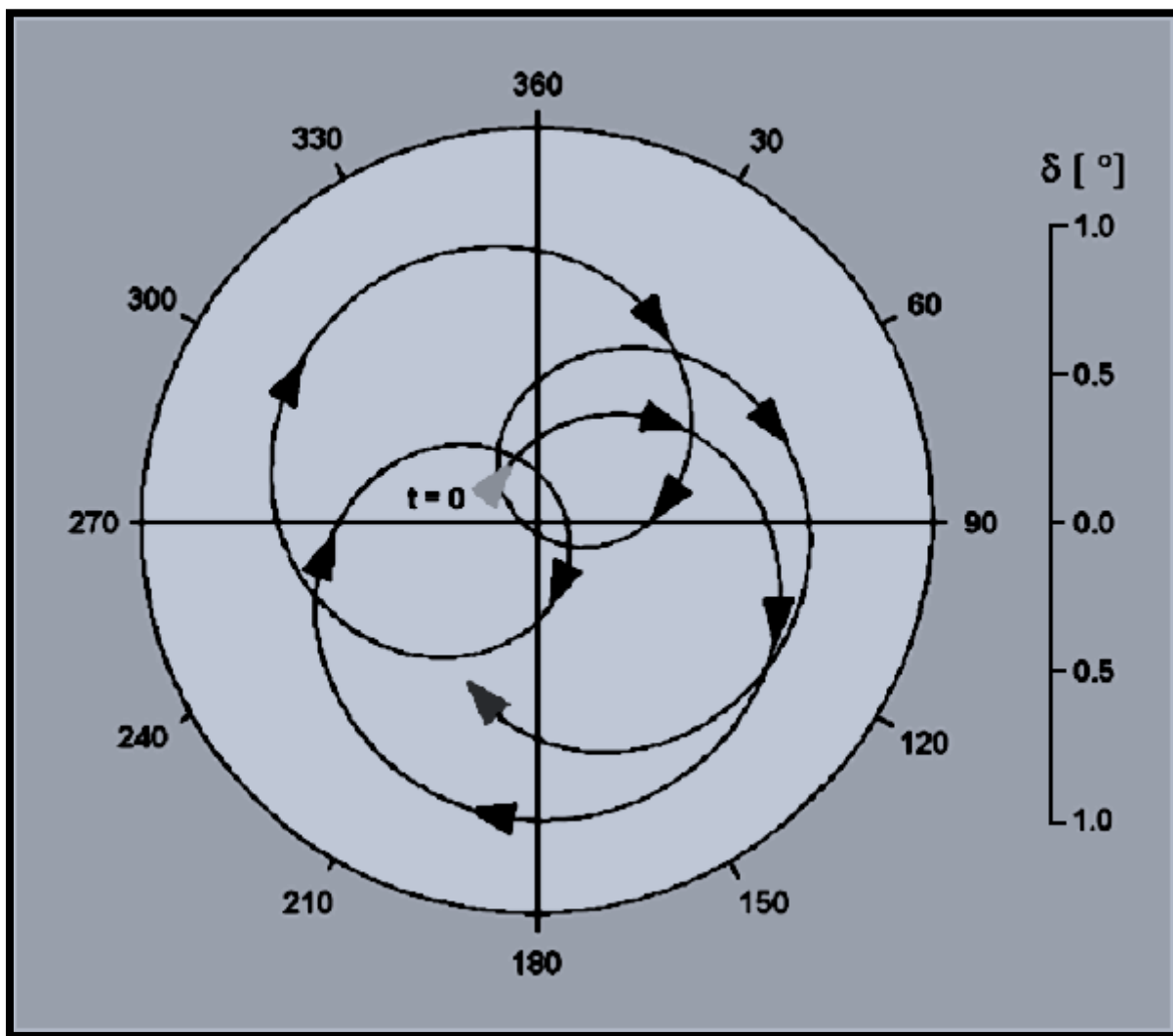
These modulations of aerodynamic angle of attack during each coning cycle produce small differential rightward-acting increments in the aerodynamic overturning moment experienced by the bullet which are centered upon the BDC and TDC positions of the bullet during each coning cycle.

In physics these are termed "torque impulses," and each one pushes the angular momentum vector of the right-hand spinning bullet horizontally rightward without affecting its spin-rate.

Both, the overturning moment vector  $\mathbf{M}$  and its differential torque impulse vector  $\Delta\mathbf{M}$  point rightward for the bullet at its BDC location. While the vector  $\mathbf{M}$  itself points leftward at TDC, its *negative* differential torque vector  $\Delta\mathbf{M}$  is *positive rightward* as well at TDC.

The 175.16-grain M118LR 30-caliber bullet used as an example here is experiencing its 88<sup>th</sup> coning half-cycle when it reaches the target distance of 1,000 yards. The reinforcing cumulative effect of these rightward torque impulses occurring twice per coning cycle is the mechanism by which gravity causes the slowly increasing rightward Yaw of Repose attitude bias of the flying bullet.

The epicyclic motion of the spin-axis direction of a typical right-hand spinning rifle bullet is shown below for the first hundred yards, or so, of its flight. The gyroscopic stability  $S_g$  of this bullet at launch is about 1.33



General form of the Yaw of Repose, as described by BRL.

$$\delta_p = - \left( \frac{8I_x \varpi}{\rho \pi d^3 CM_a V_w^4} \right) V_w \times \frac{dV_w}{dt}$$

$\delta_p$  = Yaw of Repose vector

$CM_a$  = Aerodynamic Overturning Moment Coefficient

$I_x$  = Axial Moment of Inertia

$\varpi$  = Bullet Spin Rate

$\rho$  = Air Density

$d$  = Bullet Diameter

$V_w$  = Velocity with respect of Wind

If the 3-dimensional mean trajectory of a rifle bullet in nearly horizontal flat firing is projected down onto a horizontal plane, the rightward deviation  $\beta_T$  of its tangent direction from the firing azimuth essentially defines this Yaw of Repose angle  $\beta_R$  throughout the flight, except for an even smaller horizontal tracking error  $\epsilon_H$  as the trajectory curves to the right *following* (but lagging behind) the slightly larger Yaw of Repose angle  $\beta_R$ :

$$\beta_R = \beta_T + \epsilon_H \quad (1)$$

We will formulate a good approximation for  $\beta_T$  as an aid in formulating  $\beta_R$  accurately. The horizontal tracking error angle  $\epsilon_H$  is *inherently non-negative* ( $\epsilon_H \geq 0$ ) for right-hand spinning bullets.

The Yaw of Repose has two effects on the trajectory of the projectile: 1) it produces a lateral lift-force that results in a the projectile drifting rightward (for a right-hand spinning projectile); and 2) it increases the total drag due to a small additional yaw drag component.

The additional lift is of a very small magnitude, and the accompanying additional yaw drag component is an even smaller second-order term; thus, it is omitted. Any aerodynamic lift is always accompanied by some increase in aerodynamic drag.

The horizontal spin-drift **SD** which we observe in long-range shooting is due to a horizontally acting aerodynamic *lift force* attributable to the increasing Yaw of Repose attitude angle  $\beta_R$  of the coning rifle bullet.

We will use the principles of linear aeroballistics in formulating the Yaw of Repose and its resulting Spin-Drift.

Detailed analyses of PRODAS 6-DoF simulation runs show that in flat firing the magnitude of the spin-drift **SD** in any given simulated firing is, beyond the first 150 yards or so, *nearly equal* to some invariant scale factor **ScF** of about **1.2 to 2.4 percent**, more or less, times the bullet's *drop from the projected bore axis*:

$$\mathbf{SD}(t) = -\mathbf{ScF} * \mathbf{DROP}(t) \quad (2)$$

In other words, the horizontal spin-drift trajectory looks just like a small fraction **ScF** of the vertical trajectory rotated 90 degrees about the extended axis of the bore with both curvatures ultimately caused by the same gravitational effect. The ratio of drift to drop rapidly approaches **ScF** as a limit beyond the first 150 yards of the bullet's flight.

We must formulate the scale factor **ScF** so that it can be accurately evaluated for any given bullet type and firing conditions.

Then using **Eq. 2**, we need only an accurate determination of the bullet's drop from the bore axis at the target distance to calculate an accurate spin-drift at any long-range target.

Existing 3-DoF "point mass" trajectory programs specialize in the accurate calculation of this bullet drop at the target distance in any firing conditions.

## The Horizontal Tangent Angle

The instantaneous tangent to the horizontal-plane projection of the *mean trajectory* forms the angle  $\beta_T(t)$  to an **X**-axis in that horizontal plane which defines the launch azimuth of the fired bullet. The mean trajectory of the bullet is the 3-dimensional path which would be followed by the CG of the bullet if it were not coning about that mean trajectory.

This horizontal tangent angle  $\beta_T(t)$  is always defined by the horizontal projection of the bullet's mean velocity vector, but these mean velocity components are not calculated in our available PRODAS reports. The bullet's instantaneous cross-track velocity components are modulated by the helical coning motion of the CG of the bullet in flight.

Another important use for this horizontal tangent angle function  $\beta_T(t)$  in ballistics is in plotting the horizontal yaw-attitude of the spin-axis of the bullet in the “wind axes” pitch-versus-yaw plots long used by ballisticians.

Just as the pitch coordinate data for the bullet's spin-axis direction is corrected by subtracting out the total change since launch  $\Delta\Phi_{Total}(t) = \Phi(t) - \Phi(0)$  in the vertical-plane flight path angle  $\Phi$  before plotting the pitch data for a simulated flight, the total change since launch in the mean trajectory's horizontal yaw-angle  $\beta_T(t)$  should also be subtracted out before plotting of the bullet's yaw attitude data in the interest of logical consistency.

In this way, the origin of the wind axes plots could truly be defined (horizontally as well as vertically) as the instantaneous **+V** direction of the bullet's *mean trajectory*. The horizontal tracking error angle  $\epsilon_H(t)$  would remain in the plotted yaw attitude values instead of the entire yaw-of-repose angle  $\beta_R(t)$ .

Because the Scale Factor **ScF** in Eq. 2 is essentially invariant over time **t** and distance **X(t)** at long ranges, we can evaluate the trajectory's horizontal-plane tangent angle  $\beta_T(t)$  directly from the bullet's vertical-plane **DROP** data in suitable distance units at any time **t** during its flight:

$$\beta_T(t) = dSD/dX = -ScF*[d(DROP)/dt]/V(t) \approx -ScF*[\Phi(t) - \Phi(0)] \quad (3)$$

where  $V(t) \approx dX/dt$  in flat firing.

If the scale factor **ScF** is known, this expression allows calculation of the bullet's horizontal tangent angle from either the ratio of its vertical drop rate to its forward velocity at any point during its flight or from the total change in the vertical-plane flight path angle  $\Phi$  since launch.

The “epicyclic swerve” modulation of the bullet's **DROP(t)** data rapidly fades to an insignificant fraction of the bullet's total **DROP** distance from the axis of the bore as the flight progresses.

The differential **DROP-rate** and remaining velocity **V(t)** are readily found from any simulated flight data. Calculation of the invariant Scale Factor **ScF** for any particular flight trajectory is discussed later in this paper.

Alternatively, one could evaluate  $\beta_T(t)$  directly in the horizontal plane. As the bullet drifts horizontally due solely to spin-drift **SD(t)**, the intersection point of the tangent to the bullet's mean trajectory in the horizontal plane with the **X**-axis moves forward in the **+X** direction, but at a slower velocity than the forward velocity of the bullet itself.

If we assume a continually increasing curvature of the horizontal trajectory so that this velocity ratio varies exponentially with range **X(t)**, we can estimate  $\beta_T$ , the dominant portion of  $\beta_R$ , as:

$$\beta_T \approx \text{TAN}(\beta_T) = \text{SD}(t) / \{X(t) * 0.825 * \exp[-0.925 * X(t) / X(\text{max})]\} \quad (4)$$

This hand-fitted estimator function agrees quite well with  $\beta_T(t)$  angular values extracted from available trajectories generated by PRODAS 6-DoF simulations for our particular long-range bullet by ratioing an extracted **V<sub>R</sub>(t)** to **V(t)** for each millisecond of the PRODAS trajectory reports.

The horizontally rightward velocity component data **V<sub>R</sub>(t)** is extracted by applying a smoothing difference operator to the PRODAS “no wind, no Coriolis” drift data converted into linear distance units.

Comparing the two functions for each millisecond over the **1.6923-second** simulated flight time to 1,000 yards yields a mean difference of **1.12 micro-radians** with a population standard deviation of **0.0514 milliradians**.

Extraction of the small rightward horizontal velocity **V<sub>R</sub>** from the trajectory drift data is complicated by the superimposed epicyclic swerving of the CG of the bullet which accounts for most or the variance between these two functions.

The two approaches in **Eq. 3** and **Eq. 4** for evaluating  $\beta_T(t)$  agree reasonably closely for PRODAS data as the epicyclic swerve modulations in **SD(t)** and **DROP(t)** fade in significance with ongoing flight time **t**.

We need to formulate these approximations for  $\beta_T(t)$  so that it can be used as a reasonableness check on calculations of  $\beta_R(t)$  which is not itself reported by PRODAS.

We will eventually need an accurate formulation for  $\beta_R(t)$  in order to calculate the Scale Factor **ScF** and thence the spin-drift **SD(t)** for other rifle bullet trajectories without relying upon 6-DoF simulation data.

## PRODAS Simulated Flight Data

In this paper we will use as our example bullet the US Army's 30-caliber 175.16-grain "M118LR" bullet as loaded in their M118LR Special Ball (7.62x51 mm NATO) long-range sniper and match ammunition.

We do this because we have several PRODAS 6-DoF simulation runs on hand from 2011 for this 7.62 mm NATO ammunition, reporting the linear ballistic results (including spin-drift) for each millisecond of its **1.6923-second** total simulated flight time to **1000 yards**.

The bullet weight actually used in these PRODAS runs is **175.16 grains**. The simulated firing conditions are 1) flat firing, 2) standard sea-level ICAO atmosphere, 3) no wind, 4) no Coliolis effect calculated, 5) muzzle velocity of **2600.07 feet per second**, and 6) barrel twist is right-handed at **11.5 inches per turn**.

The "no-wind" and "no-Coliolis" conditions assure that the rightward spin-drift **SD** is the only secular horizontal "bullet drift" being computed by PRODAS.

However, the PRODAS reported drift and drop data necessarily includes the oscillating horizontal component of the bullet's helical coning motion about its mean trajectory throughout its simulated flight. We also have PRODAS runs available for this same bullet fired through constant left and right 10 MPH crosswinds as well as left-hand twist runs in each of the three constant wind conditions.



We will show that in flat firing the continual downward arc of the flight path angle  $\Phi$  due to gravity causes repeated rightward differential aerodynamic torque impulses  $\Delta \mathbf{M}$  centered about the extreme top-dead-center (TDC) and bottom-dead-center (BDC) positions of the CG of the bullet in its coning motion.

These double-rate yaw attitude-changing horizontal torque impulses cause the forward-pointing angular momentum vector  $\mathbf{L}$  of the right-hand spinning bullet to shift horizontally evermore rightward during its flight. In light of Coning Theory, we should more precisely say the bullet's *coning axis* drifts horizontally rightward in its yaw attitude throughout the flight.

Ballisticians term this accumulating yaw-attitude bias the “Yaw of Repose” angle  $\beta_R$  of the flying bullet and classically formulate it from calculus as [Eq.10.83 in McCoy's MEB]:

$$\beta_R = \mathbf{P} \cdot \mathbf{G} / M \quad (5)$$

This expression is the particular solution for the differential Equations of Motion determining each trajectory in terms of the classic aeroballistic auxiliary parameters:

$$\mathbf{P} = (I_x / I_y) \cdot \mathbf{p} \cdot d / V = (\omega_1 + \omega_2) \cdot d / V \quad (6)$$

$$\mathbf{G} = g \cdot d \cdot \cos(\Phi) / V^2 \approx g \cdot d / V^2 \quad (7)$$

$$\mathbf{M} = (m \cdot d^2 / I_y) \cdot [\rho \cdot S \cdot d / (2 \cdot m)] \cdot C M \alpha = (\omega_1 + \omega_2) \cdot \omega_2 \cdot d^2 / V^2 \quad (8)$$

after converting each classic auxiliary parameter from dimensionless arc-length-rates into the time-rate units used in our analyses of flat-firing a spin-stabilized rifle bullet.

The change-of-variables in Eq. 6 uses one of the gyroscopic relationships from Tri-Cyclic Theory [Dr. John D. Nicolaides, 1953] that:

$$(I_x / I_y) \cdot \mathbf{p} = (I_x / I_y) \cdot \boldsymbol{\omega} = \omega_1 + \omega_2 \quad (9)$$

McCoy defines the canonical spin-rate  $\mathbf{p}$  of the bullet as used here to be a circular frequency given in radians per second. The bullet's spin-rate  $\mathbf{p}$  is sometimes given elsewhere in aeroballistics in units of revolutions per second (or hertz), or is sometimes given in radians per foot of bullet travel, or even in radians per caliber  $d$  of bullet travel.

To avoid this confusion we use the more conventional symbol  $\boldsymbol{\omega}$  here for the circular frequency of the spin-rate of the bullet given in radians per second. We also use the symbol  $f$  for frequencies in hertz.

The change of variables in Eq. 8 uses the fundamental magnitude relationship from Coning Theory that:

$$(\rho \cdot S \cdot V^2 / 2) \cdot d \cdot C M \alpha = L \cdot \omega_2 = (I_x \cdot \dot{\omega}) \cdot \omega_2 \quad (10)$$

as well as the Tri-Cyclic relation in **Eq. 9** again. Note that this Coning Theory relationship implies that the slow-mode coning rate  $\omega_2$  in radians per second is always given by  $(M/P) \cdot V/d$  in terms of the dimensionless aeroballistic auxiliary parameters **M** and **P**.

With these changes of variables, the classic formulation for the Yaw of Repose angle  $\beta_R$  reduces to:

$$\beta_R = g / [\omega_2(t) \cdot V(t)] = g / [2\pi \cdot f_2(t) \cdot V(t)] \quad (11)$$

While this formulation for  $\beta_R$  is classic, it *does not inherently yield zero at  $t = 0$* , and it is about a factor of  $\pi$  too small at long ranges when compared to  $\beta_T$  as formulated above [**Eq. 1** and **Eq. 3**]. The magnitude of  $\beta_R$  must always exceed that of  $\beta_T$ .

Let us say the flight path angle  $\Phi$  of the bullet's trajectory changes downward by a small decrement  $\Delta\Phi$  due solely to the pull of gravity during one-half of the period  $T_2$  of a particular coning cycle. As a continuous variable in flight time  $t$ , this angular decrement  $\Delta\Phi(t) = 0.0$  at  $t = 0$  by definition.

In flat firing, the small decrement  $\Delta\Phi$  in the nearly horizontal flight path angle  $\Phi(t)$  during the time interval  $T_2/2$  of a particular half-coning cycle can be expressed as:

$$\Delta\Phi(t) \approx \text{TAN}(\Delta\Phi) = -(g \cdot T_2) / [2 \cdot V(t)] = -g / [2 \cdot f_2(t) \cdot V(t)] = -\pi \cdot g / [\omega_2(t) \cdot V(t)] \quad (12)$$

where  $f_2(t)$  is the instantaneous coning rate, or gyroscopic precession rate, of the bullet given in revolutions per second, or hertz. [Here we are ignoring the significant cross-bore-axis (upward) component of the real bullet's ballistic drag force  $F_D$  in interest of formulating a simple **SD** estimator. This oversimplification will be explained later.]

Comparing our version of the classic formulation for the steady-state Yaw of Repose  $\beta_R(t)$  in **Eq. 11** with the change in flight path angle  $\Delta\Phi(t)$  due solely to gravity *per half-coning cycle* above, we note that:

$$\beta_R(t) = (-1/\pi) \cdot \Delta\Phi(t) \quad (13)$$

Thus our formulation in **Eq. 12** above for  $\Delta\Phi(t)$ , the change in flight path angle  $\Phi$  per half-coning cycle  $T_2/2$  due to gravity, which *does* inherently equal **zero** at  $t = 0$ , actually looks like a more suitable formulation for  $\beta_R(t)$  than does the classic form.

We will now investigate the aerodynamic and gyroscopic causes of  $\beta_R(t)$  so that we can formulate its value at any time  $t$  during the flight of any rifle bullet.

At each extreme vertical location, TDC and BDC, the coning bullet experiences a peak rate of differential change in its aerodynamic overturning moment vector **M** due to this differential change  $\Delta\Phi$  in its vertical-direction (upward airflow) aerodynamic angle-of-

attack. Each of these two differential torque impulse vectors  $\Delta \mathbf{M}$  points *horizontally rightward* as seen from behind the right-hand spinning bullet.

Here these two differential torque *impulse* vectors  $\Delta \mathbf{M}$  are to be evaluated by integrating the differential torque over each (upper or lower) half of the coning period  $T_2$ , giving them units of torque multiplied by time which correspond with the units of angular momentum.

Owing to the increased aerodynamic angle-of-attack of the apparent wind experienced by the bullet at its BDC position, the differential torque impulse  $\Delta \mathbf{M}$  at BDC is *inherently positive rightward* (as seen from behind), temporarily increasing the overturning moment  $\mathbf{M}$  acting upon the bullet at this BDC location in its coning motion.

While the overturning moment vector  $\mathbf{M}$  itself points *leftward* at the TDC position of the bullet, the differential torque impulse vector  $\Delta \mathbf{M}$  is *inherently negative* due to the *reduced* aerodynamic angle-of-attack experienced by the coning bullet at that upper location, and so the differential torque impulse vector  $\Delta \mathbf{M}$  itself points *positive rightward*, once again, at TDC.

Thus, the alternating sequences of TDC and BDC differential torque impulses are *mutually reinforcing* throughout the bullet's flight.

Recall that in Coning Theory the spin-axis of the bullet is pointing maximally *upward* when the CG of the bullet is at its BDC position in any coning cycle; i.e., its pitch attitude is a relative maximum during that coning cycle.

As formulated in linear aeroballistics, the instantaneous magnitude  $\{\mathbf{M}\}$  of the overturning moment  $\mathbf{M}$  at time  $t$  is:

$$\{\mathbf{M}\} = q * S * d * \sin[\alpha(t)] * C_M \alpha$$

Where

$q = (\rho/2) * V^2$  = Dynamic Pressure in lbf/square foot.

$\rho$  = Density of the atmosphere = **0.0764742 lbm/cubic foot** for the standard sea-level ICAO atmosphere used here. This value of the density  $\rho$  must be divided by the acceleration of gravity  $g = 32.174$  **feet per second per second** to convert its units into proper density units, mass (**slugs**) per cubic foot.

$V$  = Airspeed of the bullet in feet/second.

$S$  = Reference (frontal) area of the bullet at the base of its ogive in square feet =  $(\pi/4) * d^2$ .

$d$  = Diameter of the bullet in feet.

$\alpha(t)$  = Coning angle (and coning angle-of-attack) of the bullet in radians at any time  $t$  during its flight.

**CM $\alpha$**  = Dimensionless overturning moment coefficient in linear aeroballistics theory.

Here we are ignoring the fast-mode gyroscopic nutation  $\omega_1(t)$  of the bullet's spin-axis for several reasons: 1) It does not normally move the CG of the bullet by any measurable amount, 2) Its aeroballistic effects tend to average out to zero rather rapidly, and 3) It rapidly damps to insignificance for most rifle bullets after any flight disturbance.

As the flat-firing trajectory of the coning bullet, flying essentially horizontally near the **X**-axis (with  $\Phi \approx 0.0$  and with its coning axis aligned into the approaching windstream), arcs downward due to gravity, the aerodynamic angle-of-attack  $\alpha(t)$  increases by the magnitude of  $\Delta\Phi$  at its BDC location in this coning cycle.

The **cosines** of the coning angle  $\alpha(t) < 5.7$  degrees, the flight path angle  $\Phi$ , and the small change in flight path angle  $\Delta\Phi$  all remain essentially equal **1.00**. From trigonometry, the peak magnitude  $\{\Delta M\}_{PEAK}$  of this differential overturning torque  $\Delta M$  with the bullet at its BDC location can be expressed as:

$$\text{SIN}(\alpha + \Delta\Phi) = \text{SIN}(\alpha) * \text{COS}(\Delta\Phi) + \text{COS}(\alpha) * \text{SIN}(\Delta\Phi) \approx \text{SIN}(\alpha) + \text{SIN}(\Delta\Phi)$$

$$M + \{\Delta M\}_{PEAK} = q * S * d * \text{SIN}(\alpha + \Delta\Phi) * CM\alpha \approx M + q * S * d * \text{SIN}(\Delta\Phi) * CM\alpha$$

$$\{\Delta M\}_{PEAK} = q * S * d * \text{SIN}(\Delta\Phi) * CM\alpha \quad (14)$$

This expression can also be well approximated as:

$$\{\Delta M\}_{PEAK} = q * S * d * (\Delta\Phi) * CM\alpha \quad (15)$$

The instantaneous vertical-direction aerodynamic angle-of-attack is actually the vector sum of three small angles in complex wind-axes coordinates (ignoring the fast-mode  $\omega_1$  motion):

1. Vertical (pitch) component of the slow-mode coning angle,  $\alpha(t) * \text{COS}(\omega_2 * t + \xi_0)$
2. Downward change in flight path angle  $\Delta\Phi$ , and
3. Very small vertical-direction tracking error angle  $\epsilon_v$  (upward in wind-axes plots). This vertical-direction tracking error angle  $\epsilon_v$  is termed the “pitch-of-repose” by McCoy.

The primary overturning moment **M** is due to (1.) the coning angle-of-attack  $\alpha(t)$ . This rotating torque vector **M** produces the slow-mode circular coning motion of the CG of the bullet at the coning rate  $\omega_2(t)$  of the bullet as a gyroscopic precession of the bullet's spin-axis.

Examination of several different PRODAS runs shows that even for a *dynamically stable* bullet with any early coning motion fully damped down,  $\alpha(t)$  always exceeds  $\Delta\Phi$  by some margin all the way to maximum supersonic range and beyond.

From Coning Theory, the vertical component of the complex coning angle  $\alpha(t)$  is the “pitch angle”  $\varphi(t)$  given by the *real part* of the complex  $\alpha(t)$ , again neglecting the fast-mode motion:

$$\varphi(t) = \text{Re}[\alpha(t)] = \{\alpha(t)\} \cdot \cos(\omega_2 \cdot t + \xi_0) \quad (16)$$

Whenever  $\{\alpha(t)\} \gg \Delta\Phi$ , only the portion of  $\varphi(t)$  equal in magnitude to  $\Delta\Phi$  produces the overturning moment impulse  $\Delta M$  which drives the spin-axis of the bullet rightward giving rise to the bullet’s Yaw of Repose angle  $\beta_R$ , and the overturning moment impulses at BDC and TDC can be modeled as having the same form.

The excess of  $\varphi(t)$  over  $\Delta\Phi$  goes toward enlarging the coning angle  $\alpha(t)$ , counteracting any frictional aerodynamic damping of that slow-mode coning motion of the bullet.

The instantaneous differential overturning moment  $\{\Delta M\}$  is then due to the vertical-direction differential aerodynamic angle-of-attack  $\Delta\Phi(t) \cdot \cos(\omega_2 \cdot t + \xi_0)$ .

This modulation at the coning-rate  $\omega_2$  looks like a full-wave-rectified sine wave over each coning cycle. The time-average over each quarter wave is just  $2/\pi$  times the peak value.

The average value of  $\Delta\Phi$  itself over each half-coning cycle is just  $\Delta\Phi/2$  because the flight path angle  $\Phi$  varies almost linearly over the small interval  $T_2/2$ . Averaged over the top or bottom one-half of a coning cycle, the average effective angle-of-attack is then  $(2/\pi) \cdot \Delta\Phi/2 = \Delta\Phi/\pi$ .

The vector sum of (2)  $\Delta\Phi$  and (3)  $\epsilon_v$  varies only gradually with ongoing time-of-flight  $t$ . The *magnitudes* of these two small angles *sum* to an average vertical-direction aerodynamic angle-of-attack which drives the coning-axis direction *continually downward* according to Coning Theory, tracking (but lagging behind) the downward-curving trajectory.

The time-integrated *torque impulse*  $\Delta M$  centered at TDC or BDC must equal the differential torque due to the time-average  $\Delta\Phi/\pi$  of the modulated aerodynamic angle-of-attack multiplied by the total time interval  $T_2/2$  for each half-coning cycle. The interval  $T_2$  increases gradually as the coning rate  $\omega_2$  slows throughout the flight.

The effective differential torque impulse  $\Delta M$  integrated over a particular *half-coning cycle* thus becomes:

$$\Delta M = (T_2/2) \cdot q \cdot S \cdot d \cdot (\Delta\Phi/\pi) \cdot C M \alpha \quad (17)$$

Substituting the unsigned *magnitude* of the first expression for  $\Delta\Phi$  from Eq. 12 yields:

$$\Delta M = (1/\pi) \cdot g \cdot (T_2/2)^2 \cdot q \cdot S \cdot d \cdot C M \alpha / V(t) \quad (18)$$

This differential torque impulse  $\Delta M$  has units of lbf-feet-seconds which can be converted into slug-feet squared per second, a proper set of units for angular momentum.

The right-hand spinning bullet alters its pointing direction *rightward* in gyroscopic reaction to each of these two differential torque impulses  $\Delta \mathbf{M}$  during each coning cycle. However, it does so in an unusual way.

When a constant-magnitude, rotating overturning moment vector  $\mathbf{M}$  is applied to a spinning gyroscope, its spin-axis direction soon begins moving in precession and nutation in reaction to that steadily rotating torque vector.

However, the first motion of its spin-axis is always in the direction of the eccentric force producing the overturning moment  $\mathbf{M}$  while those epicyclic motions are getting started.

For the spinning bullet, the eccentric force is the total aerodynamic force  $\mathbf{F}$  acting through the aerodynamic center-of-pressure CP of the bullet at any instant during its flight. For spin-stabilized rifle bullets, the CP is nearly always located forward of the CG along the spin-axis of the bullet.

In response to each small torque *impulse*  $\Delta \mathbf{M}$ , the spin-axis of our bullet moves *initially rightward*, but each impulse ceases well before any vertically upward or downward movement of the spin-axis can become established.

When the torque impulse vector  $\Delta \mathbf{M}$  is expressed in the same units as the angular momentum vector  $\mathbf{L}$  of the spinning bullet, having physical dimensions of mass times length squared over time, their *direct vector sum* defines the resulting angular momentum  $\mathbf{L}$  of the spinning bullet after the torque impulse has been applied.

For a right-hand spinning bullet the angular momentum vector  $\mathbf{L}$  points forward along its spin-axis. Here, since  $\Delta \mathbf{M}$  is always acting perpendicularly to  $\mathbf{L}$ , the *direction* of the angular momentum vector  $\mathbf{L}$  is shifted rightward by an incremental angular amount (in radians) which we term  $\Delta \beta_R$ , but its *magnitude* remains unchanged.

Of course, the nose of the right-hand spinning rifle bullet in stable supersonic flight always points essentially in the direction of its angular momentum vector  $\mathbf{L}$ .

The incremental increase  $\Delta \beta_R$  in the Yaw of Repose angle  $\beta_R$  during each *half coning cycle* is thus:

$$\Delta \beta_R \approx \text{TAN}(\Delta \beta_R) = \{\Delta \mathbf{M}\}/\{\mathbf{L}\} = (1/\pi) * g * (T_2/2)^2 * q * S * d * C_M \alpha / [L * V(t)] \quad (19)$$

Recalling **Eq. 22** from the Coning Theory paper, we note that the right-hand side of **Eq. 19** above contains the fundamental expression from Coning Theory for determining the *magnitude* of the circular coning rate  $\omega_2(t)$  for a spin-stabilized bullet coning at non-zero angles of attack  $\alpha$ :

$$\omega_2 = q * S * d * C_M \alpha / L \quad (\alpha, L \neq 0)$$

Due to the acceleration of gravity, the coning angle  $\alpha(t)$  cannot be **zero** in flat firing except perhaps very briefly at  $t = 0$ , where this magnitude relationship still holds true.

After this change of variables,

$$\Delta\beta_R = (1/\pi) * g * (T_2^2) * \omega_2(t) / [4 * V(t)] \quad (20)$$

This change of variables is critically important in formulating an analytical calculation of  $\beta_R$  because it simultaneously eliminates from the formulation both the overturning moment coefficient  $CM\alpha$  and the angular momentum  $L$  of the bullet, each of which is difficult to calculate for a new bullet.

The coning rate  $\omega_2(t)$  is more readily obtainable from Tri-Cyclic Theory.

Also recall that by definition  $T_2^2 = 1/(f_2)^2 = 4\pi^2/\omega_2^2$ . After this substitution we have:

$$\Delta\beta_R = \pi * g * [\omega_2(t) * V(t)] = g / [2 * f_2(t) * V(t)] = g * T_2 / [2 * V(t)] = -\Delta\Phi \quad (21)$$

While this expression is dimensionless, the increment in the Yaw of Repose angle  $\Delta\beta_R$  for each half-coning-cycle  $T_2/2$  is calculated here in radians. The proper algebraic sign depends upon coordinate system conventions and the sense of the bullet's spin-rate.

In Linear Aeroballistics theory, the instantaneous aerodynamic lift-force driving the spin-drift  $SD$  of the bullet horizontally rightward from the  $X$ -axis is linearly proportional to the aerodynamic angle-of-attack for the very small Yaw of Repose angle  $\beta_R$ .

Thus, the linear dependence of  $\Delta\beta_R$  upon  $\Delta\Phi$  shown in Eq. 21 explains the remarkable similarity in shape of the horizontal-plane and vertical-plane projections of the bullet's "no wind, no Coriolis" mean trajectory in 3-space.

When  $\alpha(t) \gtrsim \delta \approx \Delta\Phi$ , as in most "constant wind" 6-DoF simulations, the average torque impulses  $\Delta M$  are no longer equal at BDC and TDC. In fact,  $\Delta M(BDC) \gtrsim \Delta M(TDC)$ , and their combined average effect would be slightly smaller (by about 5 percent) than these estimates here yielding a maximal Yaw of Repose angle.

The Yaw of Repose angle  $\beta_R(t)$  can be found by *summing* the increments  $\Delta\beta_R$  divided by  $T_2/2$  for each *half coning cycle* which has occurred from  $t = 0$  to time  $t$ , starting with  $\beta_R(0)$  equal **zero**.

Using the data from "no wind, no Coriolis" PRODAS reports, yields  $\beta_R(1.430 \text{ sec}) = 0.67208$  milliradians, which exceeds our fitted value of  $\beta_T(1.430 \text{ sec}) = 0.61019$  mrad by 10.14 percent.

We term this difference,  $\beta_R(t) - \beta_T(t)$ , the *horizontal tracking error angle*  $\epsilon_H(t)$ . We are comparing these angles here at  $t = 1.430$  seconds after launch when this M118LR bullet has



slowed to Mach 1.20 or **1340 feet per second** at **888.5 yards** downrange in these simulated firing conditions.

Using the PRODAS-calculated velocity and coning-rate data, our adjusted version of the classic formulation of the Yaw of Repose yields  $\beta_R(1.430 \text{ sec}) = \pi \cdot P \cdot G / M = 0.70633$  **milliradians**, which exceeds our fitted value of  $\beta_T(1.430 \text{ sec}) = 0.61019$  **mrاد** by **15.76 percent** for the horizontal tracking error angle  $\epsilon_H$ .

We believe this adjusted classic formulation for  $\beta_R$  better matches the case for a significantly coning bullet than for this particular minimally coning PRODAS trajectory.



In the absence of having 6-DoF simulation data available, we could approximate the Yaw of Repose angle  $\beta_R(t)$  by assigning readily integrable (closed form) continuous functions of time  $t$  to represent the variables  $\omega_2(t)$  and  $V(t)$  in Eq. 21 so that we could then approximate this *summing* operation by performing the definite integration of  $\Delta\beta_R(t)$  over time from  $t = 0$  to time  $t$  and dividing the integrated result by the total time interval  $t$ :

$$\beta_R(t) = (2\pi * g / t) \int [\omega_2(t) * V(t)]^{-1} dt \quad (22)$$

Here the extra factor of 2 in this expression for  $\beta_R(t)$  in Eq. 22 versus the expression for  $\Delta\beta_R(t)$  in Eq. 21 is due to integrating  $\Delta\Phi(t)$  continuously rather than using its *average value*  $\Delta\Phi/2$  over each half-coning cycle.

Note that the size of the Yaw of Repose angle  $\beta_R(t)$  whenever  $\alpha(t) \gg \Delta\Phi$  depends only on the velocity  $V(t)$  and coning rate  $\omega_2(t)$  of the bullet as functions of time. In particular,  $\beta_R(t)$  in this formulation is independent of the coning angle  $\alpha(t)$  itself in this analysis.

Since the spin-drift displacement  $SD(t)$  is caused directly by this Yaw of Repose angle  $\beta_R(t)$  as an aerodynamic lift effect, evaluation of the spin-drift  $SD(t)$  does not require detailed knowledge of the bullet's coning angle  $\alpha(t)$ . This independence of  $\beta_R(t)$  is significant because the coning angle  $\alpha(t)$  is a *free variable* in Coning Theory and is thus difficult to evaluate analytically except in special cases.

If we calculate the values of  $\omega_2(t)$  and  $V(t)$  at  $t = 0$  and at a much later flight time  $t = T$ , and we assume for approximation purposes that each function decays exponentially with time  $t$ , then the definite integral for  $\beta_R(t)$  can be expressed as:

$$\beta_R(t) = \{2\pi * g / [\omega_2(0) * V(0) * T]\} \int \exp[-(k\omega + k_v) * t / T] dt \quad (23)$$

with

$$k\omega = \ln[(\omega_2(T) / \omega_2(0))]$$

$$\omega_2(t) = \omega_2(0) * \exp[k\omega * t / T] \quad (24)$$

$$k_v = \ln[V(T) / V(0)]$$

$$V(t) = V(0) * \exp[k_v * t / T] \quad (25)$$

Here we are using  $t/T$  as a dimensionless canonical variable in the exponential decay expressions and as a dummy variable in the (summing) integration.

After the definite integration from  $t = 0$  to  $t = T$ , the expression for  $\beta_R(t)$  is:

$$\beta_R(t) = \{-2\pi * g / [\omega_2(0) * V(0) * (k\omega + k_v)]\} * \{\exp[-(k\omega + k_v) * t / T] - 1\} \quad (26)$$

Note that  $\beta_R(0) = 0.00$  as we require here.

Our “no wind anywhere” PRODAS runs for this M118LR bullet show that  $V(t)$  slows from an initial velocity of **2600.07 feet per second** to **1340 FPS** (Mach 1.20) at **888.5 yards** downrange with a time-of-flight ( $T$ ) of **1.430 seconds**, and the coning rate  $\omega_2(t)$  of the bullet slows from  **$2\pi*45.57$  radians per second** to  **$2\pi*17.00$  radians per second** over this same interval  $T$ .

The Yaw of Repose  $\beta_R(T)$  at  $T = 1.430$  seconds, and at **888.5 yards** downrange, would then be calculated as:

$$k\omega = \ln[(\omega_2(T)/\omega_2(0))] = -0.98604$$

$$k_V = \ln[V(T)/V(0)] = -0.66328$$

$$\beta_R(1.430 \text{ sec}) = [1.6464 \times 10^{-4}] * [4.2034] = 0.69206 \text{ mrad} = 0.039652 \text{ degrees} \quad (27)$$

While PRODAS does not report  $\beta_R$ , this small **0.040-degree** angle is not unreasonable for this bullet at **888.5 yards** downrange.

A smoothed value of **0.5040 milliradians** (or **0.02888 degrees**) can be directly calculated for the *tangent angle*  $\beta_T$  at 888.5 yards into a “no wind” PRODAS simulated flight by ratioing an extracted horizontally rightward velocity  $V_R(t)$  to the forward velocity  $V(t)$  of the bullet at  **$t=1.430$  seconds**.

However, this velocity ratio is very sensitive to the ongoing epicyclic swerving motion included in the PRODAS Drift reports, and its smoothed value probably should be somewhat larger here at  **$t = 1.430$  seconds**.

Our fitted algorithm, mentioned above, for estimating the tangent angle  $\beta_T$  at 888.5 yards yields **0.610186 milliradians** or **0.034961 degrees**. This would indicate a reasonable horizontal tracking error angle  $\epsilon_H$  of **0.08187 milliradians**, or **13.4 percent** of  $\beta_T$  at that point in the flight.

This closed-form integration yields a value of  $\beta_R$  about midway between our numerically-integrated value and the adjusted classic value of  $\beta_R$  as calculated from the same PRODAS data at  **$t = T$** .

We will use this closed-form algorithm (**Eq. 26**) for estimating the Yaw of Repose angle  $\beta_R(t)$  without relying upon any 6-DoF simulation data in formulating the spin-drift  **$SD(t)$**  of any rifle bullet at long ranges.

Think of these incremental yaw-attitude changes as occurring twice per coning cycle at the TDC and BDC positions of the coning bullet throughout the flight.

The double-coning-rate sequence of small torque impulses  $\Delta \mathbf{M}$  produces a reinforcing chain of these “first motions” which gradually shifts the coning-axis direction of the spinning bullet evermore rightward.

The initial Yaw of Repose at bullet launch  $\beta_R(0)$  must be **zero** by definition.

These calculations serve to validate our analysis of the basic causes of the Yaw of Repose.

## Analysis of the Spin Drift

The horizontally rightward spin-drift  $SD(t)$  of the trajectory is caused by a net horizontal aerodynamic lift-force attributable to this small, but ever increasing, rightward Yaw of Repose angular bias  $\beta_R(t)$  in the yaw-attitude of the coning-axis of the spinning bullet.

The pointing direction of the bullet's coning axis quickly tracks each of these small changes in the approaching apparent wind direction within one half of a coning cycle, just as with any other type of wind change.

As the horizontal projection of the *mean trajectory* traced by the *mean CG* of the bullet gradually accelerates rightward with this spin-drift  $SD(t)$ , its tangent  $+V$  direction defining the origin of wind-axes plots should properly drift slowly rightward also, *following* (but lagging) the increasing Yaw of Repose attitude angle  $\beta_R(t)$  of the bullet.

We formulated this tangent angle  $\beta_T(t)$  earlier. Logically, only the horizontal tracking error angle  $\epsilon_H(t) = \beta_R(t) - \beta_T(t) \geq 0$  should appear in these wind-axes plots.

In formulating the effective net (time-averaged) aerodynamic lift-force accelerating the CG of the coning bullet rightward, we must consider the coning modulation of the aerodynamic effect as the CG of the bullet moves throughout its circular coning cycle.

Here the modulation is horizontally left-to-right, and the effect being modulated is an aerodynamic lift force.

However, for the uniformly coning rifle bullet, analysis of the modulation of this lift force can be greatly simplified by making use of Coning Theory.

We can express the average effective aerodynamic lift-force on the coning bullet arising from the Yaw of Repose angle  $\beta_R(t)$  as if the bullet were *not* coning, but simply flying with the spin-axis always aligned with the attitude of its coning axis  $[\alpha(t) = 0]$ . After all, it is the attitude of that coning axis which properly defines this Yaw of Repose angle.

The actual average aerodynamic angle of attack in a coordinate system moving with the bullet is just the tracking error angle  $\epsilon_H(t)$ . The lift-force attributable to this  $\epsilon_H(t)$  angle of attack keeps increasing the rightward curvature of the mean trajectory. In earth-fixed coordinates, the average horizontal angle of attack driving the mean trajectory away from the X-axis is  $\epsilon_H(t) + \beta_T(t) = \beta_R(t)$ .

The magnitude of the small net rightward aerodynamic lift force  $\{F_L\}_R$  attributable to the rightward yaw attitude bias  $\beta_R(t)$  of the coning axis is given in linear aeroballistics as:

$$\{F_L\}_R = q \cdot S \cdot CL_{\beta} \cdot \sin[\beta_T(t) + \epsilon_H(t)] - q \cdot S \cdot CD \cdot \sin[\beta_T(t)]$$

Or

$$\{F_L\}_R \approx q(t)*S*CL_{\beta}(t)*\beta_R(t) - q(t)*S*CD(t)*\beta_T(t) \quad (29)$$

The small rightward aerodynamic lift force acting horizontally on the bullet is actually counteracted partially by an even smaller cross-bore component of the bullet's significant aerodynamic drag force  $F_D$  given by  $q*S*CD*\beta_T(t)$ .

Here the coefficient of lift  $CL_{\beta}(t)$  and coefficient of drag  $CD(t)$  are evaluated for the very small aerodynamic angle-of-attack  $\beta_R(t)$ . However, they still vary with the Mach-speed of the slowing bullet. The dynamic pressure  $q(t)$  also reduces with the square of its airspeed  $V(t)$  as the bullet slows.

This small rightward horizontal force  $\{F_L\}_R$  acting on a bullet of mass  $m$  for one half the period  $T_2$  of each coning cycle produces a rightward horizontal bullet velocity increment  $\Delta V_R$  given here in **feet per second per half-coning cycle** as:

$$\Delta V_R = \{F_L\}_R * T_2 / (2 * m) = \{F_L\}_R / [2 * m * f_2(t)] = (\pi / m) * \{F_L\}_R / \omega_2(t) \quad (30)$$

Where  $m$  is the mass of the bullet expressed in **slugs**. Here,  $m = 175.16 / (7000 * g) = 0.00077774$  **slugs**. We are using  $g = 32.174$  **feet per second per second** for the standard effective “acceleration of gravity” on or near the surface of our rotating earth.

These rightward velocity increments  $\Delta V_R$  accumulate (sum) from **zero** at  $t = 0$  for each *half coning cycle* which occurs from launch to time  $t$  to form the horizontally rightward velocity  $V_R(t)$  of the CG of the bullet which is caused aerodynamically by the Yaw of Repose angle  $\beta_R(t)$ .

The incremental rightward horizontal spin-drift of the bullet,  $\Delta SD$  in feet, during one particular half-coning cycle  $T_2/2$  is then:

$$\Delta SD(t) = V_R(t) * T_2 / 2 = V_R(t) / [2 * f_2(t)] = \pi * V_R(t) / \omega_2(t) \quad (31)$$

The horizontal spin-drift  $SD(t)$  at time  $t$  is then found by *summing* these incremental displacements  $\Delta SD(t)$  for each *half coning cycle* starting with **zero** at  $t = 0$ . Our subject M118LR bullet experiences **87 half-coning cycles** during its flight to **1000 yards**.

Numerical integration of  $\Delta SD(t)$  using PRODAS data for each millisecond of the simulated “no wind” flight yields  $SD(1.6923 \text{ sec}) = 9.7019$  **inches**. PRODAS itself calculates a total drift of **9.5407 inches** at **1000 yards**. The PRODAS drift includes the horizontal component of the minimal coning motion of the spinning bullet.

This level of agreement verifies our aeroballistic analysis of the causes of spin-drift.

## Analytic Calculation of the Spin Drift at the Target

If we formulate a reliable estimation of the scale factor **ScF** for any given bullet in any given firing conditions, this scale factor **ScF** can then be used together with a reliably calculated value of that bullet's *DROP from the bore axis at the target* to calculate analytically the spin-drift **SD(t)** at the target for any given rifle bullet in any firing conditions according to **Eq. 2**:

$$\text{SD}(t) = -\text{ScF} * \text{DROP}(t) \quad (2)$$

We know from examination of available “constant crosswind” PRODAS 6-DoF simulations that the scale factor **ScF** needs to be **0.0219685** for the “constant no-wind” runs, and **0.0222219 (or 1.154 percent larger)** for the “constant 10 MPH crosswind” runs for this example M118LR bullet fired in these simulated conditions to **1000 yards**.

The unrealistic “constant no-wind” PRODAS case represents the *minimum possible SD(t)* values for this bullet fired at this muzzle velocity and spin-rate in this atmosphere and flying with the minimum possible coning motion throughout its flight.

Most *dynamically stable* rifle bullets fired outdoors at long ranges will likely suffer only the minimal **1.154 percent** increased “constant 10 MPH crosswind” type of spin-drift **SD(t)** at long ranges.

However, a *dynamically unstable* rifle bullet, such as the infamous mid-range 30-caliber 168-grain Sierra International, might experience about **5 percent** greater spin-drift (**SD**) when fired to long ranges (up to 800 meters) through *any* non-zero crosswinds.

## Estimating the Ratio of Second Moments of Inertia for Rifle Bullets

We need an accurate estimation of the ratio  $I_y/I_x$  for our rifle bullet so that we can find the coning rate  $\omega_2(t)$  of that bullet at any time  $t$  during its flight from Tri-Cyclic Theory.

We will use the *reference diameter* for our subject bullet  $d(\text{in feet}) = 1.00$  calibers as the distance metric throughout these calculations.

By definition, the second moment of inertia of the bullet about any of its principal axes can be expressed as:

$$\begin{aligned} I_x &= m \cdot k_x^2 \\ I_y &= m \cdot k_y^2 \end{aligned} \tag{32}$$

where

$m$  = mass of the bullet

$k_x$  = *radius of gyration* about the spin-axis of the bullet (a principal axis of inertia), and

$k_y$  = *radius of gyration* about any transverse principal axis through the CG of the bullet.

The *radius of gyration* for any mass distributed about any possible axis of rotation is the radial distance from that axis at which all of that mass could be considered to be concentrated into a thin shell for purposes of calculating the second moment of that mass distribution about that axis.

So, the ratio  $I_y/I_x$  becomes:

$$I_y/I_x = (k_y/k_x)^2 \tag{33}$$

We have accurately measured  $k_x$  for a typical long-range rifle bullet to be **0.238352 calibers**, and we believe this  $k_x$  value (in calibers) to be reasonably invariant across the population of the better long-range rifle bullet designs.

We analytically “sliced” a uniform-density long-range rifle bullet design into many thin discs of equal thickness, each perpendicular to the spin-axis, and then calculated the average radius over the length of the solid bullet as **0.337081 calibers**.

We then multiplied this average outside radius by one half the square root of two (or **0.707107**, the radius of gyration ratio for each thin, uniform-density disc about a central perpendicular axis) to arrive at the radius of gyration  $k_x = \mathbf{0.238352}$  calibers about the spin-axis of this representative long-range rifle bullet.

We estimate  $k_y$  for any long-range rifle bullet based on several measurable bullet parameters and the features often found in long-range rifle bullet designs: 1) presence of a

Meplat, 2) presence of a Boat-Tail, and 3) presence of a Rear Driving-Band. We do not include the presence of hollow cavities mainly because of the difficulties in measuring and specifying them.

We calculate a Length-Effect **LE** upon **ky** based on the length **L** of the bullet in calibers:

$$LE = 0.50 * \text{SQRT}(L^2/3 + 0.25) \quad (34)$$

This expression calculates the transverse radius of gyration **ky** for a uniform-density “short cylinder” of **1.0-caliber** diameter and **L calibers** in length about a transverse axis through its midpoint CG.

We calculate a Meplat-Effect **ME** upon **ky** as:

$$ME = 21.0 * [LOC_{CG} + (3 * LN + LFN)/4] * VOL_{NC} / VOL_{BUL} \quad (35)$$

where

**21.0** = Empirically determined weight factor

**LOC<sub>CG</sub>** = CG location (in calibers) behind the base of the bullet’s ogive

**LN** = Measured Length of the bullet’s nose in calibers

**LFN** = Full length of bullet’s complete ogive in calibers (as calculated in the CWAJ paper)

**VOL<sub>NC</sub>** =  $(-\pi/12) * (DM^2) * (LFN - LN)$  = Missing volume of truncated nose cone in cubic calibers, and

**VOL<sub>BUL</sub>** =  $(Wt \text{ of bullet in grains}) / [(2750 \text{ gr/cu. in.}) * (12*d)^3]$  = Estimated volume of the bullet in cubic calibers. Here we are using **2750 grains per cubic inch** as an estimated average density of a jacketed lead-core long-range rifle bullet.

The large relative weight factor (**21.0**) used here is to account for the disparate impact on **ky** versus **kx** caused by missing this nose-cone mass.

We calculate a Boat-Tail Effect **BTE** on **ky** as:

$$BTE = (L - LN - LBT/4 - LOC_{CG}) * VOL_{BT} / VOL_{BUL} \quad (36)$$

**LBT** = Measured length of boat-tail in calibers, and

**VOL<sub>BT</sub>** =  $(-\pi/12) * LBT * [2 - DBT * (DBT + 1)]$  = Missing volume due to boat-tail, and

**DBT** = Measured diameter of base of boat-tail in calibers.

A relative weight factor of unity is assigned to this Boat-Tail Effect in this analysis.

We calculate a Rear Driving-Band Effect **RDBE** upon **ky** as:



$$RDBE = 23.0 * (LOC_{DB} - LOC_{CG}) * VOL_{DB} / VOL_{BUL} \quad (37)$$

Where

**23.0** = Empirical weight factor

**LOC<sub>DB</sub>** = Measured midpoint location (in calibers) of the rear driving-band behind the base of the ogive

**VOL<sub>DB</sub>** =  $(-\pi/4) * WDB * (DDB^2 - 1)$  = Extra volume of driving-band in cubic calibers, with

**WDB** = Measured effective width of driving-band in calibers

**DDB** = Measured outside diameter of driving-band in calibers. **DDB = 1.00 calibers** if no driving-band is present.

This correction is also *negative* for a rear driving-band bullet design because it is missing a thin outer shell of mass everywhere forward of its driving band. The empirical weight factor accounts for the ratio of these “shell” masses and their disparate **I<sub>y</sub>/I<sub>x</sub>** effects, predominantly a disproportionate reduction in **k<sub>x</sub>** versus **k<sub>y</sub>**.

The radius of gyration **k<sub>y</sub>** about a transverse principal axis (in calibers) is then found by summing:

$$k_y = 0.679897 * (LE + ME + BTE + RDBE) \quad (38)$$

The initial factor **0.679897** adjusts **k<sub>y</sub>** for a typically tapered “rifle bullet” shape and for its typically offset CG location from its mid-length location as distinct from the uniform-diameter “short cylinder” model used above in developing the Length-Effect **LE** in this physical analysis.

The missing volume of the Meplat-Effect **ME** is modeled as a small right-circular cone of height **h** with its mass concentrated at a CG located at **h/4** on its axis.

The missing boat-tail volume is modeled as a 1.0-caliber cylinder of length **LBT** minus the frustum of a conical boat-tail with the CG of the missing volume located at **h/4** from its (thicker) aft end. The Rear Driving-Band is modeled as a thin-walled cylinder.

Any low-density polymer or aluminum “tip” used in the design of a long-range rifle bullet should be ignored in making your bullet measurements. The possible presence of hollow cavities in any bullet designs has not yet been separately addressed because of the expected difficulty of measuring and specifying these internal cavities.

The estimated value of **I<sub>y</sub>/I<sub>x</sub>** is then calculated for this bullet in accordance with **Eq. 33** as:

$$I_y/I_x = (k_y/0.238352)^2 \quad (39)$$

This estimator calculates the second moment ratio  **$I_y/I_x$**  as **9.067260** for the 30-caliber 175.16-grain M118LR bullet, which matches exactly (because we selected it as typical of all long-range bullets) the  **$I_y/I_x$**  ratio which can be extracted very precisely from our available PRODAS simulation data using that bullet.

In other words, we recovered the exact  **$I_y/I_x$**  ratio which was calculated by the proprietary PRODAS pre-processor for this particular type of bullet. PRODAS is well regarded world-wide for the fidelity of its flight simulations to actual measurements of the flights of fired projectiles of all types.

Applying this estimator to data for the old 30-caliber 168-grain Sierra International bullet [from McCoy, page 217 MEB] yields an  **$I_y/I_x$**  ratio of **7.4787** or **0.502 percent** greater than the value **7.4413** reported by McCoy.

Applying this estimator to a 30-caliber 173-grain monolithic brass ultra-low-drag bullet of new design yields an  **$I_y/I_x$**  ratio of **13.4623** or **0.081 percent** less than the value **13.4733** calculated by numerical integration of the mass distribution for that bullet design.

However, we did use the calculated CG location of **0.5575 calibers** behind the ogive and the known density of the brass material **2128 grains per cubic inch** for that new bullet design.

We calculated the accurate value of  **$k_x = 0.238352$  calibers** from the detailed numerical data describing that patented bullet design as well. This monolithic (solid) brass bullet design has uniform density throughout and has no hollow cavities.

## Estimating the Spin Drift Scale Factor $ScF$

With all this in mind, we formulate an estimator for an  $ScF$  value for the “constant 10 MPH crosswind” type of coning motion of our example bullet which can be duplicated for any other *dynamically stable* bullet in any likely firing conditions.

In flat firing, we can formulate the scale factor  $ScF$  in terms of the *ratio* of 1) the horizontal aerodynamic lift-force acting on the flying bullet due to its Yaw of Repose  $\beta_R$  as that bullet nears its long-range target to 2) the vertically downward-acting weight of that bullet. In this manner, we can formulate  $ScF$  for any given bullet and likely wind conditions as:

$$ScF = 1.01154 * 0.383703 * [q(t) * S] * \sin[\beta_R(t)] * CL_{\beta}(t) / Wt$$

$$ScF = 0.388132 * [q(t) * S] * \beta_R(t) * CL_{\beta}(t) / Wt \quad (40)$$

with  $Wt = 175.16/7000$  representing the weight of this M118LR bullet given in pounds-force **lbf**.

Here again, as in the earlier simplified formulation of  $\Delta\Phi$ , we are ignoring the upward force on the free-flying bullet caused by the cross-bore component of its ballistic drag force. The force offset effects of these two simplifications cancel out here in forming this ratio for evaluating  $ScF$ .

The initial constants (**0.383703** and **0.388132**) have been empirically determined from several PRODAS runs and should be the same for any *dynamically stable* rifle bullet in any firing conditions likely to be encountered in long-range shooting. The PRODAS runs show the M118LR bullet to be dynamically stable.

These PRODAS simulations, together with the classic formulation for the yaw of repose angle  $\beta_R(t)$ , indicate that the scale factor  $ScF$  might need to be increased by about **5 percent** for *dynamically unstable* bullets which will fly with significant coning angles throughout their flight when fired through *any crosswinds*.

Each of these functions of time  $t$  should be evaluated at the time  $T$  when the bullet has slowed to an airspeed of **1340 feet per second** (or approximately **Mach 1.20**, depending upon ambient conditions).

This flight time  $T$  and the flight distance at which it occurs are completely independent of the actual range to the target. The time  $T$  can be determined by using a 3-DoF point-mass trajectory calculator.

Any good long-range rifle bullet should remain safely above the turbulent transonic region at this **1340 fps** airspeed in almost any reasonable atmospheric conditions. The more “aerodynamic” of our lowest-drag long-range rifle bullet designs will not encounter

transonic buffeting until they slow to about **Mach 1.10** airspeed. The needed coefficient of lift  $CL_\beta$  is particularly difficult to estimate for any bullet in the transonic airspeed regime.

Most experienced long-range riflemen select their shooting equipment so that whenever possible their bullets impact the target at airspeeds above **Mach 1.2**

For similar best-accuracy reasons, we base our calculation of  $ScF$  upon bullet data at **1340 fps** airspeed regardless of the actual range to the intended target.

We calculate the potential drag-force  $q(T)*S$  using the calculated density  $\rho$  of the ambient atmosphere in **slugs per cubic foot**, and the airspeed  $V(T) = 1340$  feet per second:

$$q(T)*S = (\pi*d^2/4)*(\rho/2)*(1340 \text{ fps})^2 \quad (41)$$

This potential drag-force value should be about **1.1 lbf** for a **30-caliber** bullet at this airspeed depending on air density  $\rho$ . The potential drag-force at **1340 fps** varies most strongly with the square of the caliber  $d$  of the bullet (in feet).

The analytic estimate of  $\beta_R(T)$  is calculated per **Eq. 26** above with  $t = T$  and  $V(T) = 1340$  fps. With these simplifications **Eq. 26** becomes:

$$\beta_R(T) = -g*\{\exp[-(k\omega + k_v)] - 1\}/[f_2(0)*V(0)*(k\omega + k_v)] \quad (42)$$

If we know the *initial* gyroscopic stability  $S_g$  of the bullet, we can calculate the initial Stability Ratio  $R$  of its epicyclic rates  $f_1/f_2$  from:

$$R = 2*\{S_g + \text{SQRT}[S_g*(S_g - 1)]\} - 1 \quad (43)$$

The *initial* coning rate  $f_2(0)$  in hertz can then be found from Tri-Cyclic Theory as:

$$f_2(0) = V(0)/[Tw*(I_y/I_x)*(R + 1)] \quad (44)$$

where

$Tw$  = Absolute value of the Twist Rate of barrel in feet per turn.

$I_y/I_x$  = Ratio of transverse to axial second moments of inertia for this bullet as estimated above.

Substituting back into **Eq. 42**, we have:

$$\beta_R(T) = -g*Tw*(I_y/I_x)*(R + 1)*\{\exp[-(k\omega + k_v)] - 1\}/[V^2(0)*(k\omega + k_v)] \quad (45)$$

where

$V(0)$  = Launch velocity of this particular bullet in feet per second. [ $V(0)$  is assumed to exceed **Mach 2.0**]

$k\omega = -0.98604$ , as before for any long-range bullet, and

$$k_v = \ln[1340/V(0)].$$

This value  $\beta_R(T)$  in radians is the estimated Yaw of Repose angle for this bullet when it has slowed to **1340 fps**.

The  $CL_\beta(T)$  value is estimated based on an estimate of the initial  $CL_\beta(0)$  for the Mach-speed of the bullet at launch (here **Mach 2.3289**) evaluated from Robert L. McCoy's INTLIFT program for the nose-length effect, but using our own boat-tail effect lift reduction for these long-range bullets.

We multiply McCoy's nose-length estimated  $CL$  by the square root of **0.2720/BC7** for each bullet, reasoning that about half of any differing drag for bullets having higher or lower ballistic coefficients **BC7** (ballistic coefficient relative to the G7 Reference Projectile) than our example M118LR bullet is due to having a more or less effective boat-tail design.

If a more reliable **BC1** value ( $BC$  relative to the G1 Reference Projectile) is available for your rifle bullet, use the square root of **0.5460/BC1** for this estimated  $CL$  adjustment for variations in bullet drag.

The full nose-length (**LFN**) for the 30-caliber M118LR bullet is **2.5955 calibers**. The initial coefficient of lift  $CL_\beta(0)$  at this **Mach 2.3289** airspeed calculates to **2.720** using our adjusted INTLIFT estimate.

Very slightly scaling McCoy's published lift curve for the well-studied 30-caliber 168-grain Sierra International bullet to this lift coefficient **2.720** at this **Mach 2.3289** airspeed yields a coefficient of lift  $CL_\beta(T)$  of **1.877** at **Mach 1.20** in this standard ICAO atmosphere.

The bullet's coefficient of drag  $CD_0$  determines its time-rate of decay in Mach-speed. The supersonic lift-to-drag ratio  $F_L/F_D$  for any given angle-of-attack tends to be an invariant aerodynamic characteristic of each basic bullet shape.

Since the coefficients of lift and drag are highly correlated at any given Mach-speed over the population of long-range rifle bullets, the same exponential time-decay coefficient:

$$k_L = \ln[CL_\beta(T)/CL_\beta(0)] = -0.3711 \quad (46)$$

can be used in propagating the coefficients of lift  $CL_\beta(T)$  for any long-range rifle bullets of interest here.

We propagate this *initial* coefficient of lift  $CL_\beta(0)$  estimate forward to its value at time **T** as:

$$CL_\beta(T) = CL_\beta(0) * \text{EXP}[-0.3711 * (V(0)/2600 \text{ fps})^2 * (1.430 \text{ sec}/T)] \quad (47)$$

The coefficient of lift  $CL_\beta(T)$  for any very-low-drag (VLD) or ultra-low-drag (ULD) long-range rifle bullet should be smaller than **1.90** at this airspeed of **1340 fps**. Bullets designed

for lower aerodynamic drag will also produce less aerodynamic lift. Conversely, you cannot produce more lift without also increasing drag in aerodynamics.

The exponential propagation function [Eq. 47] estimates a larger fraction of the initial coefficient of lift  $CL_{\beta}(0)$  remaining at **1340 fps** airspeed when the time-of-flight **T** *to that airspeed* is increased due to firing a higher-drag bullet, but the initial velocity correction factor  $[V(0)/2600 \text{ fps}]^2$  prevents this increase when time-of-flight **T** to **1340 fps** increases simply due to firing that same bullet with a higher muzzle velocity  $V(0)$ .

That is, if the *same bullet* is fired at different muzzle velocities, its estimated coefficient of lift  $CL_{\beta}(T)$  at **1340 fps** airspeed should remain the same.

The muzzle velocity  $V(0)$  is assumed to exceed **Mach 2**. This coefficient of lift propagation yields the expected  $CL_{\beta}(T) = 1.8769$  at **1340 fps** for the M118LR bullet, and varies by less than **1 percent** over any reasonable launch speeds  $V(0)$  for this one bullet type.

The scale factor **ScF** is now calculated from Eq. 40 using the values of the time-functions at time **T** as calculated in Eq. 41, Eq. 45, and Eq. 47 above:

$$\text{ScF} = 0.388132 * [q(T) * S] * \beta_R(T) * CL_{\beta}(T) / Wt \quad (48)$$

where

**0.388132** = An empirically determined constant (from PRODAS data) for all firings of “normally coning” *dynamically stable* rifle bullets through any non-zero, “reasonably constant” (non-diabolical) crosswinds.

This constant is numerically necessary for several likely reasons, chief among them that the driving horizontal lift-force  $F_L[t, \beta_R(t)]$  is actually attributable only to the horizontal tracking error attitude angle  $\epsilon_H(t)$  instead of the entire yaw-of-repose angle  $\beta_R(t)$ .

If we might be slightly misestimating the Yaw of Repose angle  $\beta_R(T)$  or coefficient of lift  $CL_{\beta}(T)$  used here in any systematic ways for these minimal-coning-motion “constant wind” 6-DoF flight simulations, the empirically-determined initial constant factor **0.388132**, from that same PRODAS data, tends to absorb any net difference.

## Calculating the Spin Drift at the Target

The spin-drift **SD(tof)** at the *target distance* is calculated from **Eq. 2** above using the invariant Scale Factor **ScF**, as calculated in **Eq. 48** above for the bullet slowed to **1340 fps**, and the total **DROP** from the axis of the bore for the time-of-flight (**tof**) to the actual target:

$$\text{SD}(\text{tof}) = -\text{ScF} * \text{DROP}(\text{tof}) \quad (49)$$

The Spin-Drift **SD** at the target is calculated here in **Eq. 49** in the same *distance* or *angular units* in which the bullet's **DROP** from the bore axis is given. Again, the proper algebraic sign depends upon coordinate system conventions and the sense of the bullet's spin rotation.

The bullet **DROP** and time-of-flight (**tof**) to the target are accurately calculated in many existing 3-DoF point-mass trajectory propagators. After all, the accurate calculation of bullet **DROP** at the target distance is the basic figure of merit for these software aids.

The time-of-flight (**tof**) to the target is used in the Litz **SD** estimator and is nice to know even if we do not actually need it in these calculations.

If your particular trajectory propagation program does not directly output “drop from bore axis” data, you can usually “fake” it into doing so by setting your scope height equal zero, setting the angle-of-fire accurately equal to that of the anticipated shot, setting the rifle's “zero range” equal to some minimum distance (ideally zero, or perhaps 5 or 10 yards if made necessary by input limit constraints), and specifying that the trajectory calculations go out to the target's known or measured range.

In other words, we want to calculate the **DROP** from the bore axis at the target distance as if we were “bore-sighted” on that long-range target.

The smoothed spin-drift reported by PRODAS at 1000 yards for “constant zero-wind” simulations with this 175.16-grain M118LR bullet fired in these conditions is **9.5407 inches**. The spin-drift **SD** estimated via this algorithm for 1000 yards (without the factor of **1.01154** increase in **ScF**) is **9.5635 inches**.

Comparing the two results millisecond-by-millisecond, throughout the flight of **1692.3 milliseconds**, yields a mean difference of **0.0043 inches**, with a population standard deviation of **0.0207 inches**.

This level of agreement between our estimator for spin-drift for each millisecond and the PRODAS numerical simulation results is rather astonishing. The rounding error for drop and drift data in the PRODAS report format is **0.180 inches** at 1000 yards, and we are not even estimating the horizontal component of the bullet's small coning motion which is included in the PRODAS drift data.



The agreement of this spin-drift **SD** estimator with PRODAS “constant 10 MPH crosswind” runs is also about this good when the Scale Factor **ScF** includes the **1.154 percent** increase as formulated above.

This **1.154-percent-augmented** version of **ScF** in **Eq. 48** should be calculated for firing any other *dynamically stable* rifle bullets outdoors.

## Summary

- I. We fit an exponential tangent angle function  $\beta_T(t)$  to extracted velocity-ratio data from a PRODAS simulation which minimizes the epicyclic swerve complications in measuring the Yaw of Repose angle  $\beta_R(t)$ . We discovered that the spin-drift **SD** at long range is affected slightly (about **5 percent**) by the magnitude  $\alpha(t)$  of coning motion experienced by the bullet en route to the target, with consistently larger coning angles  $\alpha(t)$  producing slightly more spin-drift **SD**(t).
- II. We define the horizontal and vertical direction tracking error angles,  $\epsilon_H$  and  $\epsilon_V$  respectively, which should appear in a ballisticians’ wind-axes plots resulting from 6-DoF flight simulations. Just as with the flight path angle  $\Phi$ , the Yaw of Repose angle  $\beta_R(t)$  logically should not appear in those wind-axes plots which reference the **+V** direction of the 3-dimensional mean trajectory as their origin. We provide an analytic formulation in **Eq. 3** for  $\beta_T(t)$ , the horizontal tangent angle, which logically should be subtracted from the bullet’s spin-axis yaw attitude data before plotting.
- III. We explain the aerodynamic causes of Yaw of Repose and spin-drift and numerically verify those explanations using data from PRODAS 6-DoF simulations together with the principles of linear aeroballistics theory.
- IV. We reformulate the classic aeroballistic Yaw of Repose angle as  $\beta_R = \pi PG/M$ , which holds for a significantly coning bullet with  $\alpha(t) \gg \Delta\Phi$  throughout its flight. Furthermore,  $\beta_R(0) = 0.00$  at launch by definition. For a minimally coning bullet with  $\alpha(t) \gtrsim \Delta\Phi$ ,  $\beta_R = \beta_T + \epsilon_H \approx 0.95\pi PG/M$ .
- V. We formulated an accurate analytic estimator for the ratio  $I_y/I_x$  of the second moments of inertia for any long-range rifle bullet so that the sum of its two epicyclic rates ( $\omega_1 + \omega_2$ ) can be calculated via Tri-Cyclic Theory from the remaining circular spin-rate  $\omega$  of the bullet (in radians per second) at any time  $t$  during its flight.



- VI. We note that in flat firing the spin-drift displacement **SD** of the bullet at any long range is essentially an invariant scale factor **ScF** times the bullet's drop distance from the projected bore axis at that range. The scale factor **ScF** runs about **1.2 to 2.4 percent** for the various long-range rifle bullets in typical flat firing. That bullet's drop from the axis of the bore is accurately computed in any 3-DoF trajectory propagation program. This same scale factor **ScF** defines the ratio of the horizontal and vertical angular deviations of the tangent to the *mean trajectory* from the axis of the bore at firing time (when  $t = 0$ ). The angular deviation in the horizontal plane  $\beta_T(t)$  is always equal to the Scale Factor **ScF** times the vertical-direction deviation  $\Delta\Phi_{Total}(t) = \Phi(t) - \Phi(0)$ .
- VII. We present an analytic calculation of that invariant scale factor **ScF** so that an accurate and reliable calculation of spin-drift **SD(t)** can be computed for any long-range rifle bullet flat-fired in any likely conditions without relying upon 6-DoF simulations. This dimensionless Scale Factor **ScF** can also be used as part of a collection **K** of invariant values from Eq. 40 such that the Yaw of Repose angle  $\beta_R(t)$  can be calculated for any flight time  $t$  as:

$$\beta_R(t) = K/[V^2(t)*CL_\beta(t)] \quad (50)$$

with

$$K = (ScF/0.388132)*[Wt/(\rho*S/2)] \quad (51)$$

This formulation of  $\beta_R(t)$  is very similar to the classic formulation for  $\beta_R(t)$ , and this formulation also does not evaluate to zero at  $t = 0$ . This calculated non-zero initial yaw-of-repose attitude angle  $\beta_R(0) \approx 0.130$  milliradians is just the initial yaw attitude which would be required to produce a hypothetical horizontal lift force of **ScF\*Wt** at muzzle velocity **V(0)**. Of course, no such side-force exists at bullet launch.

## Example Calculations of $I_y/I_x$

The parameters and calculations needed to determine Yaw of Repose  $\beta_R$  and spin-drift  $SD$  are shown in tabular format below without resorting to use of information obtained from 6-DoF simulations.

Three different 30-caliber rifle bullets are selected for these parallel calculations for variety and based on availability of bullet measurements including  $I_y/I_x$  ratios. Two additional 30-caliber rifle bullets are included because they were tested in “drift firings” by Bryan Litz.

The obsolete 168-grain Sierra International bullet (for which McCoy supplies the needed data) is similar to their current improved 168-grain MatchKing (SMK). We have several PRODAS runs for the bullet used in the US Army M118LR 7.62 mm NATO Special Ball ammunition.

The 175.16-grain M118LR bullet is used as the pattern for calculating  $I_y/I_x$  for any other bullet, so its calculated  $I_y/I_x$  value is exactly that used by PRODAS. We are using the bullet shape parameters for the similar 30-caliber 175-grain Sierra MatchKing bullet in lieu of the data on the M118LR bullet until such data can be obtained.

The 173-grain solid (monolithic) brass Ultra-Low-Drag (ULD) bullet design has not yet been tested, but its numerical design description allows accurate modeling of its flight characteristics using McCoy’s aeroballistic estimators.

The data on the Berger 175-grain Open-Tip Match (OTM) Tactical bullet and their 185-grain Long Range Boat Tail (LRBT) bullets, as well as the test conditions during their 1000-yard drift firing, were taken from Bryan’s publications.

The analytic method used here for estimating the  $I_y/I_x$  ratio for rifle bullets is calibrated against the value **9.067260** used by PRODAS for the 30-caliber M118LR bullet. The pre-processor estimation of  $I_y/I_x$  used in PRODAS is proprietary, but its results are certainly well accepted in the ballistics community.

The  $I_y/I_x$  ratio of **7.4413** is published by McCoy for the old 168-grain Sierra International bullet. Our estimate of **7.4787** is **0.502 percent** larger than McCoy’s measured value.

The target  $I_y/I_x$  ratio of **13.4733** for the new monolithic brass ULD bullet was calculated by numerical integration of its elements of mass. Our estimated value here of **13.4624** is **0.081 percent** smaller than this value.

The data used here for the two Berger 30-caliber bullets selected by Bryan Litz in his drift firing experiments at 1000 yards are taken from his publications.

30-Caliber Example Bullets:	168-gr International	175-gr M118LR	173-gr ULD(SB)	175-gr Berger Tactical	185-gr Berger LR-BT
Bullet Length L (cal)	3.9800	4.0260	5.4368	4.1169	4.3929
Nose Length LN (cal)	2.2600	2.3052	2.8368	2.3701	2.5747
Diameter of Meplat DM (cal)	0.2500	0.2175	0.1000	0.1948	0.2013
Length of Boat-Tail LBT (cal)	0.5100	0.5360	0.7012	0.6331	0.5844
Diameter of Base DB (cal)	0.7645	0.8280	0.8420	0.8409	0.8182
Ratio of Ogive Generating Radii RT/R	0.9000	1.0000	0.5000	0.9000	0.9500
CG loc (cal. from ogive)	0.2000	0.3000	0.5575	0.4000	0.0000
Volume (Cu. Cal.)	2.090858	2.17797	3.004986	2.177977	2.302432
Calc. Tangent Ogive Radius RT (cal)	6.997633	6.98661	9.166594	7.177891	8.499340
Calc. Full Tangent Ogive Length LFT (cal)	2.597621	2.59549	2.986067	2.632089	2.872166
Calc. Full Conical Nose Length LFC (cal)	3.013333	2.94594	3.152000	2.943548	3.223577
Calc. Full Nose Length LFN (cal)	2.639192	2.59549	3.069033	2.663235	2.889737
ky effects:					
Length-Effect (LE)	1.175812	1.18879	1.589255	1.214452	1.292517
Meplat-Effect (ME)	-0.159206	-0.0928	-0.014669	-0.079836	-0.080889
BT-Effect (BTE)	-0.057892	-0.0403	-0.051220	-0.040878	-0.056933
Dr-Band-Effect (DBE)	0.000000	0.00000	-0.237083	0.000000	0.000000
SUM = LE+ME+BTE+DBE	0.958714	1.05563	1.286284	1.093738	1.154695
SUM/Target ky Ratios:	0.678195	0.67989	0.680173		
Target $ky=kx*\text{SQRT}(Iy/Ix)$	0.650195	0.71772	0.874896		
Target Iy/Ix	7.441300	9.067260	13.473300		
<b>Calculated Iy/Ix:</b>	<b>7.478685</b>	<b>9.06726</b>	<b>13.462337</b>	<b>9.733605</b>	<b>10.848792</b>
Iy/Ix Error:	0.037385	0.00000	-0.010963		
Percent Error:	0.502393	0.00000	-0.081369		

## Example Calculations of Spin Drift

The remaining parameters needed to calculate Yaw of Repose  $\beta_R$  and spin-drift **SD** are calculated for the five example bullets in the spreadsheet shown below. A 3-DoF trajectory program was used to compute the time-of-flight (**tof**) and flight distance to an airspeed of **1340 FPS** and **tof** to a 1000-yard target for both the 168-grain SMK bullet and the new 173-grain ULD bullet. PRODAS trajectory data was used for the 175.16-grain M118LR bullet.

The initial gyroscopic stability factor **Sg** was taken from McCoy for the old 168-grain Sierra International bullet, and **Sg** is calculated using McCoy's McGYRO program for the new 173-grain ULD bullet.

PRODAS reports the **Sg**-value for each millisecond of the flight of the M118LR bullet, but we just used their initial value. Bryan Litz gives the initial **Sg** values for the two Berger bullets used in his drift firings.

The ULD bullet is a dual-diameter design with the base of the ogive measuring **0.3002 inches** in diameter (**1.0-calibers** for this bullet design). It has a rear driving-band measuring **0.3082 inches** in diameter (or **1.02665 calibers**). The midpoint (CG) of the rear driving-band is located **1.6 calibers** behind the base of its 3-caliber secant ogive, and the width of this driving band is **0.6 calibers**.

Our calculated Yaw of Repose angles  $\beta_R$  for the first three example bullets when each has slowed to an airspeed of **1340 FPS** shows an interesting progression.

The estimated Yaw of Repose angles  $\beta_R$  of the three trajectories at the **1340 FPS** airspeed points are **0.471231 milliradians** for the obsolete 168-grain International bullet at **816 yards** downrange, and **0.693417 milliradians** at **888.5 yards** downrange for the M118LR bullet, but just **0.434382 milliradians** for the new 173-grain monolithic brass ULD bullet, and this occurs way beyond the 1000-yard target at **1457 yards** downrange.

The assumed **3200 fps** muzzle velocity of this new ULD bullet is based on firing it from a 300 Remington UltraMag cartridge. Each of the other example 30-caliber bullets is assumed to be fired from a much less powerful 7.62 mm NATO or 308 Winchester cartridge.

For comparison purposes the spin-drift **SD** at **1000 yards** is calculated in inches for each of our five example bullets using the **SD** estimator published by Bryan Litz:

$$\text{SD(inches)} = 1.25 * (\text{Sg} + 1.2) * (\text{tof})^{1.83} \quad (52)$$

Our estimates of **SD** at 1000 yards are *smaller* than Bryan's estimates for each of these five example bullets. Our estimate of spin-drift **SD** at 1000 yards for the M118LR bullet of

**9.7037 inches** matches the **SD** computed by PRODAS (**9.5407 inches**) quite closely (error: **+0.163 inches**, or **+1.708 percent**).

We expected this value to be **1.154 percent** too large because this PRODAS run is for the unrealistic “no wind anywhere” case causing absolute minimum coning motion of this numerically defined “perfect bullet.” Thus, the unexplained difference between our **SD** estimate and that computed in PRODAS is **0.554 percent**. The Litz-estimated **SD** of **10.2791 inches** for this M118LR bullet at 1000 yards exceeds the PRODAS value by **0.7156 inches**, or **+7.483 percent**.

Our estimate of **7.0152 inches** of spin-drift **SD** at 1000 yards for the old 168-grain Sierra International bullet from a 12-inch twist barrel is approximately **2.135 inches** less than the **9.15 inches** shown graphically by McCoy in **Figure 9.8** of his MEB, and is **3.348 inches** less than the **10.020 inches** calculated by the Litz estimator for this bullet.

We expected our estimate to be at least **5 percent** (or **0.458 inches**) too small for this *dynamically unstable* bullet. We cannot readily explain the remainder of this difference other than point to the unusually high coning rate  $\omega_2(t)$  of this bullet throughout its flight due to its low **ly/lx** ratio.

Our estimate of **3.1214 inches** of spin-drift **SD** at 1000 yards for the radical new 173-grain monolithic brass ULD bullet design, versus the value of **5.1696 inches** calculated by the Litz estimator for this bullet, indicates the need for our more elaborate **SD** calculation in predicting the long-range flights of current and future ultra-low-lift rifle bullets, even when fired from faster twist-rate barrels.

The Litz spin-drift estimator is closer than our estimator to reported spin-drift values for the old 168-grain Sierra International bullet and for the Berger 175-grain OTM Tactical bullet.

Our estimator seems closer for the remaining three bullets, especially for the two very-low-drag (and correspondingly very-low-lift) bullets—the brass 173-grain ULD bullet and the Berger 185-grain Long Range Boat-Tail (LR-BT) bullet.

Of course, our predictive agreement with the PRODAS calculations for the M118LR bullet is best of all. We expect that if 6-DoF simulations could be run for the other four bullets, our estimator would match those results more closely.

We also suspect that Bryan’s drift firing results would not match linear 6-DoF simulation results particularly well if they could be computed.

The aerodynamic responses of real rifle bullets are non-linear enough to affect the calculation of these small second-order effects. Real bullets are also subject to other types of aerodynamic jump phenomena in real firings—some of which might be at least partially systematic.

<b>Spin-Drift Example Calculations:</b>				Temp (Degrees-F)	37.00
	Std ICAO Atmosphere:			Rel. Humidity	70.00
Rho (Lbm/Cu. Ft.)	0.0764742	0.0764742	0.0764742	Stat Pressure (InHg)	28.9500
Rho (Slugs/Cu.Ft.)	0.002376894	0.0023768	0.002376	0.002397459	0.00239745
Mach 1.00 = a (FPS)	1116.45	1116.45	1116.45	1093.23	1093.23
<b>30-Caliber Example Bullets:</b>	<b>168-gr International</b>	<b>175.16-gr M118LR</b>	<b>173-gr ULD(SB)</b>	<b>175-gr Berger Tactical</b>	<b>185-gr Berger LR-BT</b>
Bullet Length L (cal)	3.9800	4.0260	5.4368	4.1169	4.3929
Nose Length LN (cal)	2.2600	2.3052	2.8368	2.3701	2.5747
Diameter of Meplat DM (cal)	0.2500	0.2175	0.1000	0.1948	0.2013
Length of Boat-Tail LBT (cal)	0.5100	0.5360	0.7012	0.6331	0.5844
Diameter of Base DB (cal)	0.7645	0.8280	0.8420	0.8409	0.8182
Ratio of Ogive Generating Radii RT/R	0.9000	1.0000	0.5000	0.9000	0.9500
Calc. Tangent Ogive Radius RT (cal)	6.9976	6.9866	9.1666	7.1779	8.4993
Calc. Full Tangent Ogive Length LFT (cal)	2.5976	2.5955	2.9861	2.6321	2.8722
Calc. Full Conical Nose Length LFC (cal)	3.0133	2.9459	3.1520	2.9435	3.2236
Calc. Full Nose Length LFN (cal)	2.6392	2.5955	3.0690	2.6632	2.8897
V0=Launch velocity (FPS):	2800.00	2600.07	3200.00	2660.00	2630.00
Initial Mach-Speed	2.5079	2.3289	2.8662	2.4332	2.4057
Initial B-value	2.3000	2.1032	2.6861	2.2182	2.1880
Ballistic Coef (G1 Ref)	0.4260	0.5460	0.6290	0.5060	0.5530
Ballistic Coef (G7 Ref)	0.2180	0.2720	0.3220	0.2580	0.2830
INTLIFT CL(0)	3.1015	2.7203	2.5551	2.8145	2.6189
Time T to 1340 FPS (sec)	1.2723	1.4300	2.1070	1.3972	1.5030
Range at 1340 FPS Airspeed (yards)	816.00	888.50	1457.00	881.5400	839.1000
Est CL(T) at 1340 FPS:	1.9120	1.8769	1.7447	1.8913	1.8248

Twist Rate (inches/turn, RH)	12.0000	11.5000	8.2500	10.0000	10.0000
Initial Sg	1.7400	1.9400	1.5940	2.2400	1.9100
Initial Stability Ratio (R)	4.7494	5.5808	4.1341	6.8132	5.4567
Calculated (Iy/Ix) Ratio	7.478685	9.067260	13.46233	9.7336	10.8488
Initial Coning Rate f2 (hz)	65.1188	45.4687	67.3429	41.9719	45.0549
kv=LN(1340/V(0))	-0.736950	-0.662869	-0.870481	-0.685657	-0.674314
komega+kv	-1.722990	-1.648909	-1.85652	-1.671697	-1.660354
Beta-R at time T (mrad)	0.471231	0.693417	0.434382	0.744921	0.696844
Ref. Diam. (1.0 cal. in inches):	0.3080	0.3080	0.3002	0.3080	0.3080
Frontal Area S (square feet)	0.000517403	0.0005174	0.000491	0.0005174	0.00051740
Potential Drag Force at 1340 fps (lbf)	1.1041	1.1041	1.0489	1.1041	1.1041
Bullet Weight (grains)	168.0	175.1600	173.0	175.0	185.0
Bullet Weight (lbf)	0.0240000	0.0250229	0.02471	0.02500	0.0264286
Calculated Scale Factor ScF	0.016088267	0.0222895	0.012484	0.0241500	0.02061980
DROP from Bore Axis at 1000 yds (inches)	436.0450	435.3450	250.0250	428.4970	414.8350
Time of Flight (tof) to 1000 yds (seconds)	1.7300	1.6923	1.2390	1.6870	1.6400
Remaining Velocity at 1000 yds (FPS)	1145.00	1213.99	1836.00	1197.00	1278.00
DROP from Bore Axis at 100 yds (inches)	2.5500	2.6750	2.1400	2.7950	2.8550
Calc. 100-yard SD (inches)	0.0410	0.0596	0.0267	0.0675	0.0589
<b>Calculated 1000-yd Spin-Drift (inches rightward)</b>	<b>7.0152</b>	<b>9.7036</b>	<b>3.1214</b>	<b>10.3482</b>	<b>8.5538</b>
SD from McCoy Figure 9.8	9.1500				
SD from PRODA runs		9.5407			
SD (inches) from Drift Firings				11.4000	6.7000
Litz Est. Spin-Drift (inches rightward)	10.0203	10.2791	5.1696	11.1967	9.6125
Litz Est. Spin-Drift Minus Our Calc. SD (inches)	3.0051	0.5754	2.0482	0.8485	1.0586

Sensitivity analysis was significant in studying and assessing the uncertainty in the output of our model, which can be attributed to different sources of error of the input parameters.

Sensitivity analysis is an integral part of model development and involves analytical examination of input parameters to aid in model validation and provide guidance for future research.

We used it to determine how different values of one or more independent variables, impact a particular dependent variable under a given set of conditions.

In other words, it helped us to investigate the robustness of the model predictions and to explore the impact of varying input assumptions.

We chose to set on what is known as Local (sampled) sensitivity analysis, which is derivative based (numerical or analytical). The use of this technique is the assessment of the local impact of input factors' variation on model response by concentrating on the sensitivity in vicinity of a set of factor values.

Such sensitivity is often evaluated through 2-dimensional gradients or partial derivatives of the output functions at these factor values, (the values of other input factors are kept constant) when studying the local sensitivity of a given input factor.

One of the critical objectives was to stress-test the model as well as to study its fidelity to known experimental and model-based 6-DoF runs.

Unfortunately there are many results and accompanying charts to add, but in order to make the document more manageable we've chosen to include only a pair of them, which are significant in terms of reliability of the underlying numerical algorithm.

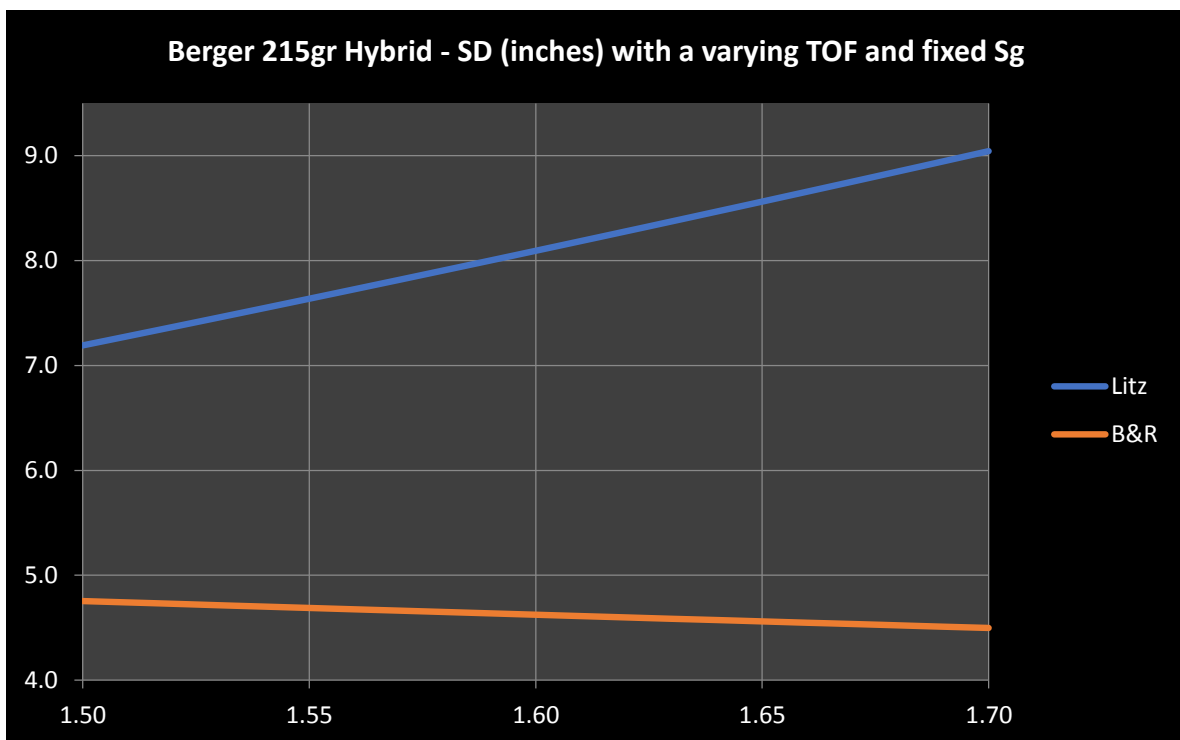
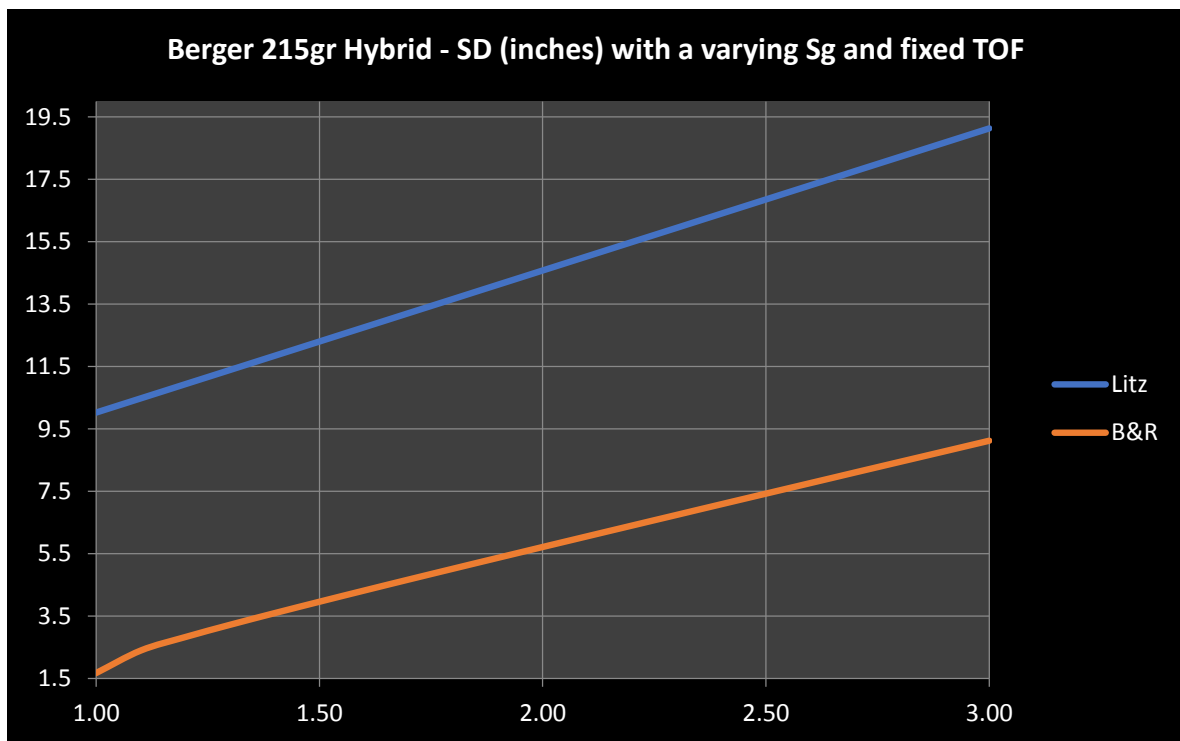
The following charts compare the outputs of three models used to estimate **SD**, namely Hornady 4-DoF, Litz and the B&R method presented in this paper.

In the case of Hornady's 4-DoF, the reader must take into consideration that three major variables, **Sg**, **DROP** and **ToF**, are different than the ones used to calculate the B&R and Litz outputs because it produces different **DROP** and **ToF** values as well as a varying **Sg**.

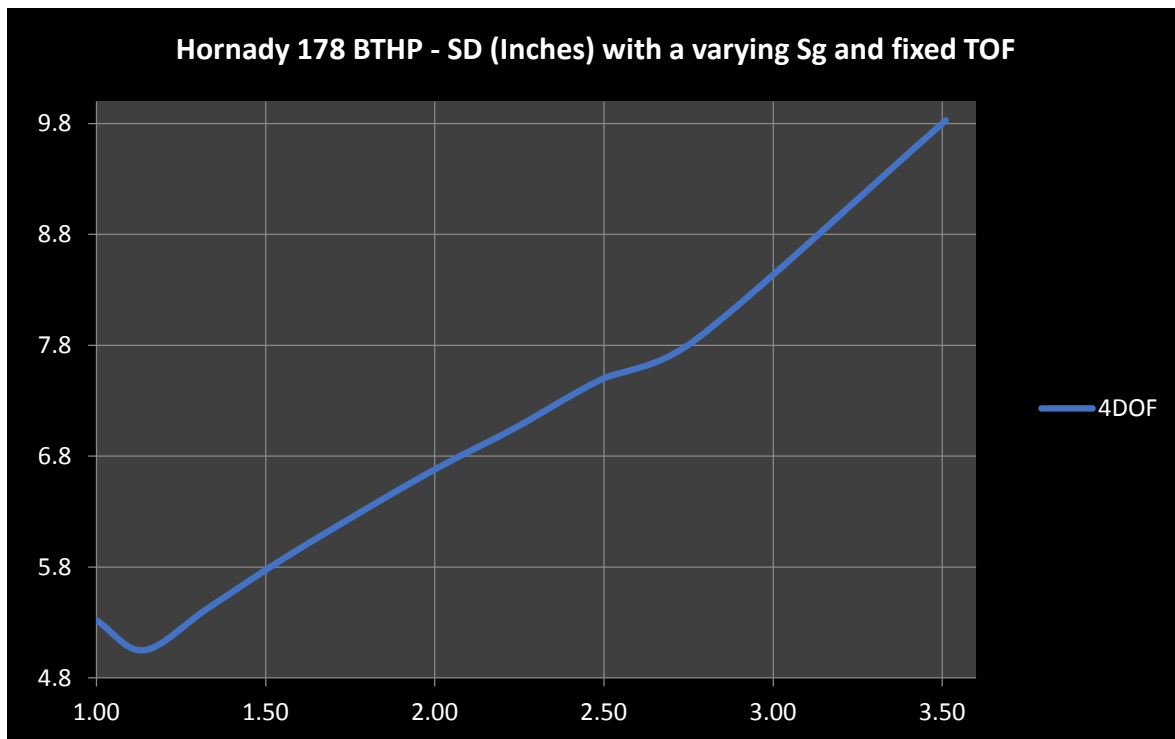
On the other hand, all **DROP** and **ToF** figures are the same for both Litz and B&R, and were calculated with a common 3-DoF Point Mass software with a fixed-muzzle-only **Sg** based on Miller' rule. Indeed neither model is intended to work with a progressively increasing static stability.



As can be easily seen, the response to a varying **Sg** with a fixed **ToF** is clearly linear for both models. Same behavior for a varying **ToF** with a fixed **Sg**. Bear in mind that the variation ranges are quite narrow, which is the normal and expected uncertainty of these inputs.



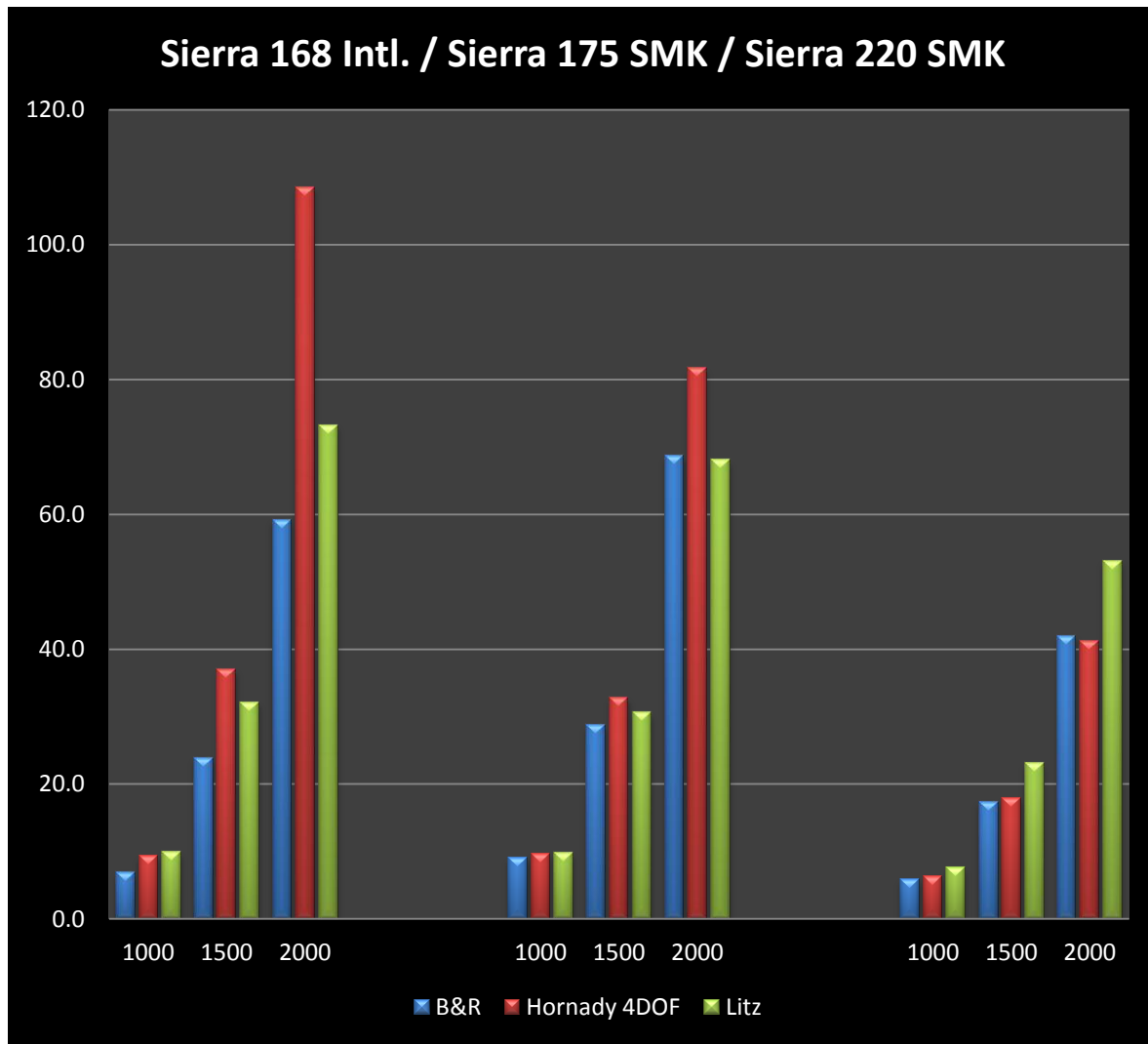
The slight decrease in **SD** with increasing **ToF** shown above for the B&R model is explained by not adjusting the velocity  $V(t)$  of the bullet as **ToF** is varied. The B&R model uses  $V(t)$  explicitly in many places.



In the 4-DoF case, the model response to a varying initial **Sg** with a fixed **ToF**, is quasi-linear and also exhibits a quite similar magnitude of the delta variation of **Sg** as the Litz and B&R models.

The 4-DoF (Modified Point-Mass, Lieske & Reiter, 1966) provides an estimate of the Yaw of Repose. This model considers the bullet rolling motion around its longitudinal axis of symmetry, called spinning motion. Therefore, this model presents four degrees of freedom: three translational coordinates for describing position and one for angular speed.

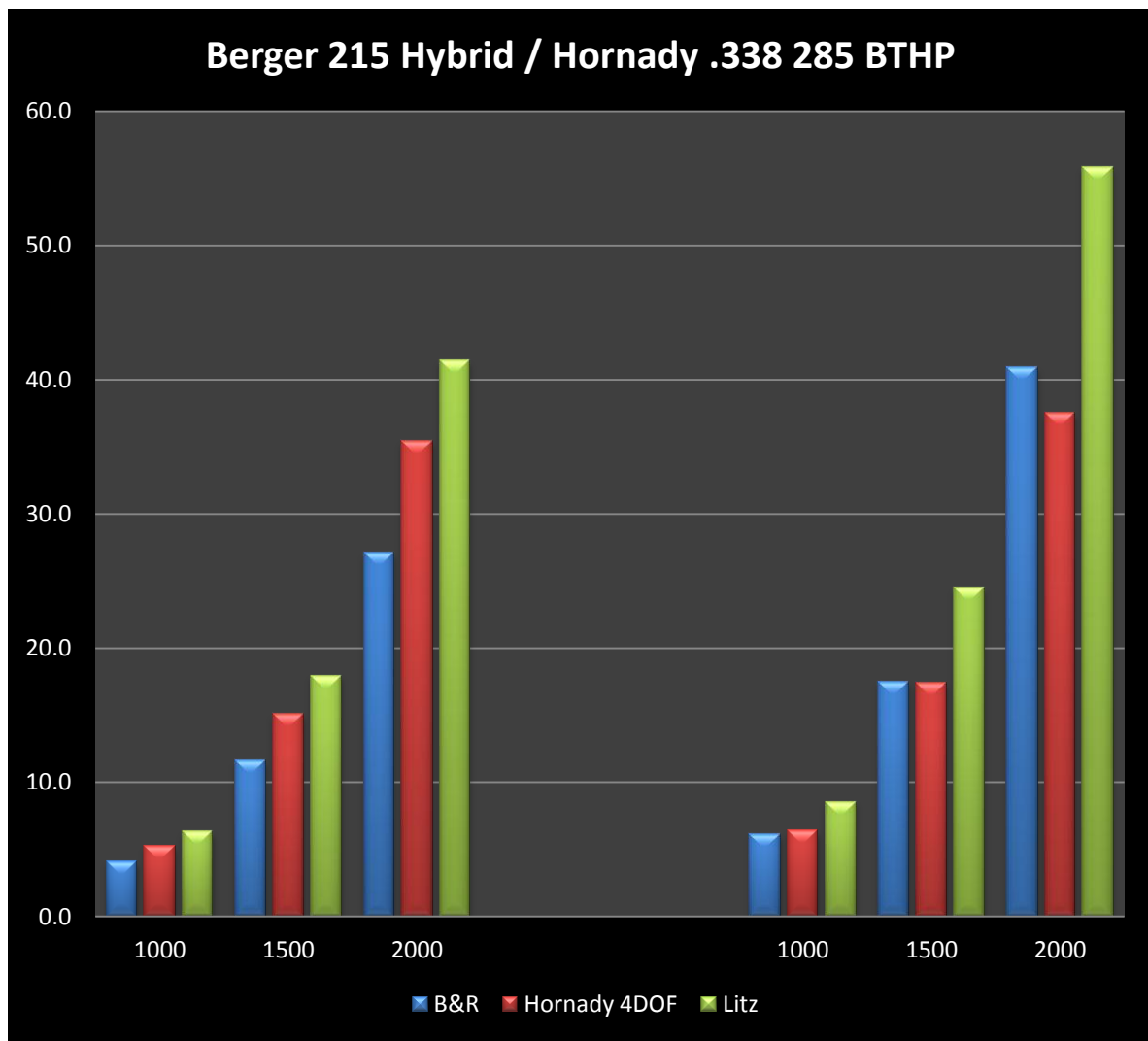
Some may argue that the underlying phenomena calls for a more elaborated multi-parameter analysis and while the concept is right, we chose to perform a single parameter analysis in order to compare to the Litz model which is a simple 2D model and as such does not relate the influence of one parameter over the other as the bullet goes down range, namely the aerodynamic coefficients.



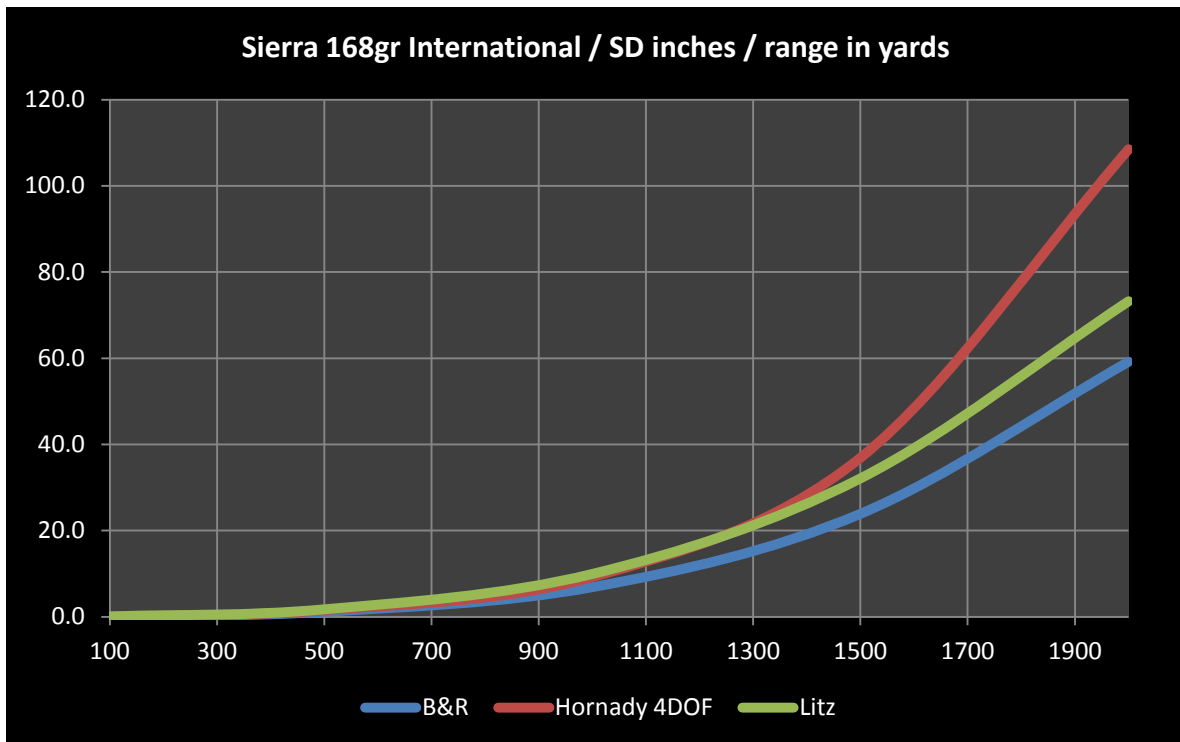
The Spin Drift is expressed inches, while each bullet is compared with the three different estimators, and grouped at 1000, 1500 and 2000 yards, which are typical ranges for Extreme Long Range shooting.

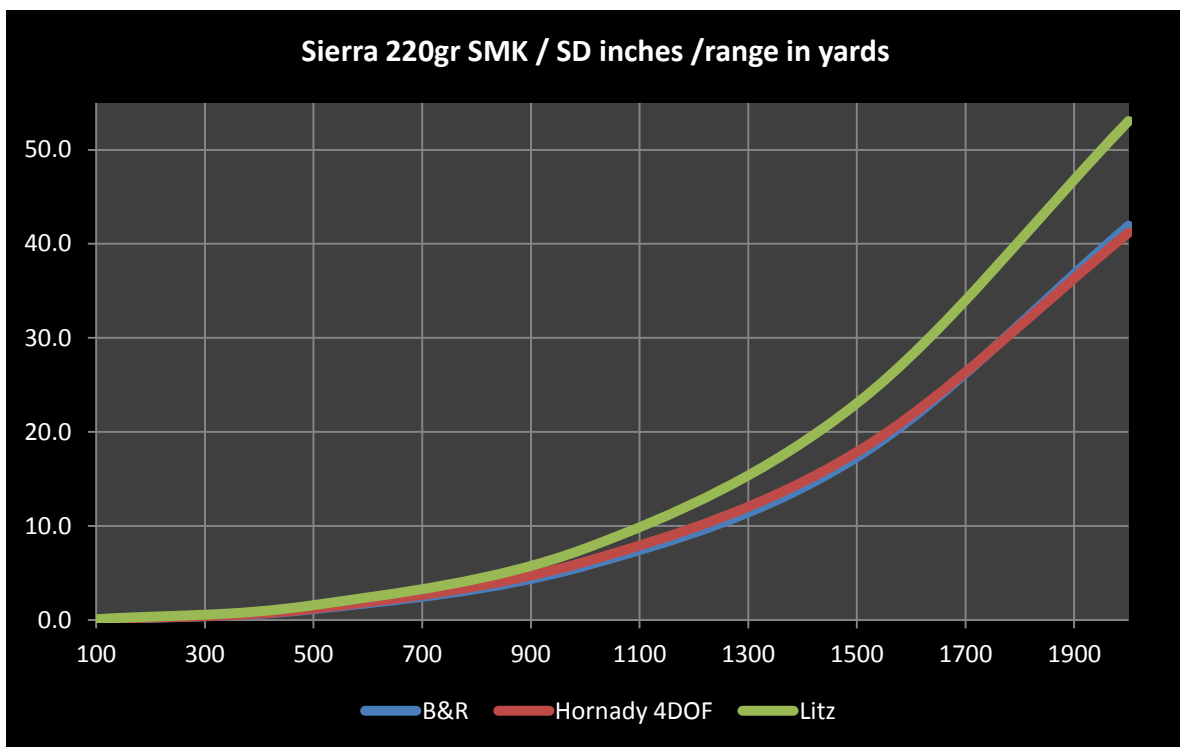
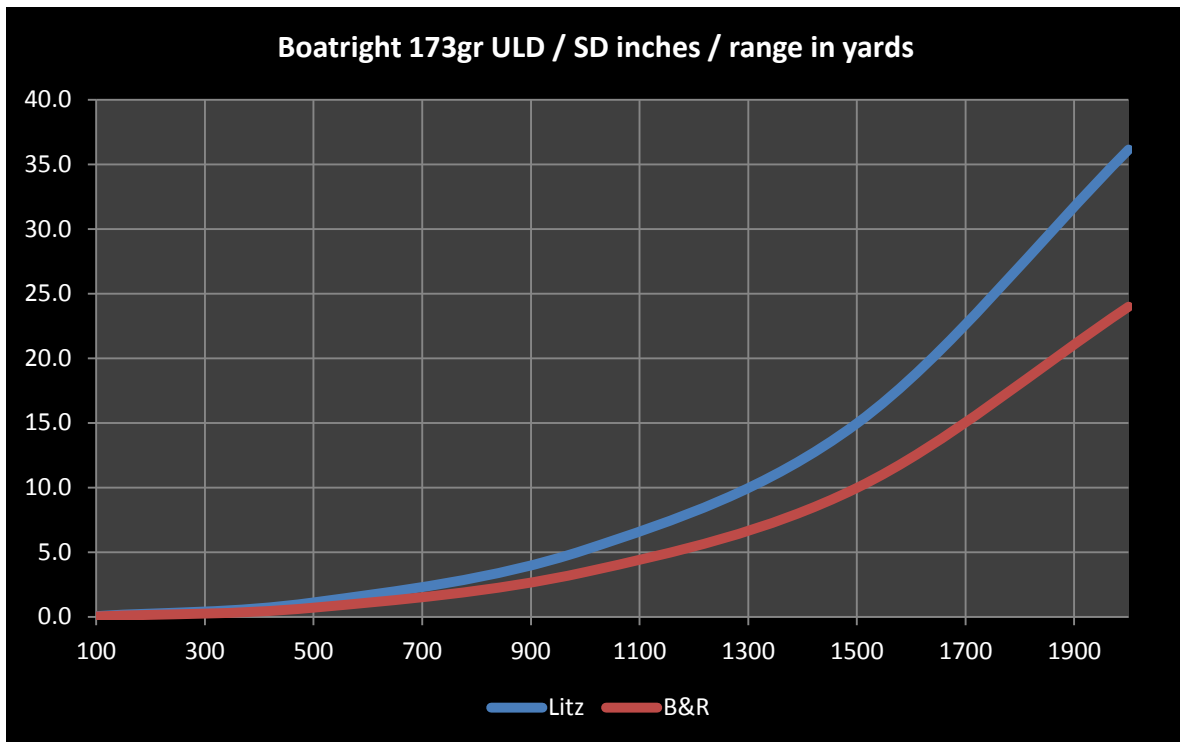
The Litz estimator does fair work, given its simple inputs, but its reliability is dictated by the underlying aerodynamics characteristics of the bullet, which are not accounted for in this simple linear approach.

Consequently, as soon as the bullet does not exhibit certain properties that cannot be encompassed by **Sg** alone, its predictive accuracy is decidedly affected. In general terms, the Litz model tends to over predict **SD** in a significant way.



As can be appreciated, as the range increases, the difference among the estimators becomes larger. The practical side of this is that the correct method is of paramount importance when dealing with Extreme Long Range shooting.





## Closing Summary

Taken together, the implications of **Eq. 8** and **Eq. 19** determine the bullet and rifle characteristics which affect the size of the horizontal spin-drift **SD(t)** which will be seen in flat firing at a long-range target.

*First*, we see from **Eq. 8** that **SD(t)** displacement is always proportional to the bullet's **DROP(t)** in distance units from the projected axis of the bore during firing.

This implies that modern lighter-weight “flat shooting” bullets fired at higher muzzle velocities **V(0)** and retaining more of that velocity farther downrange (higher ballistic coefficient, lower drag bullets) will produce much less spin-drift **SD(t)** at any target distance compared to slower, higher-drag bullets. That is, **SD(t)** is roughly proportional to time-of-flight **t** to the target distance.

*Second*, according to **Eq. 19**, the size of the scale factor **ScF**, and thence the size of the spin-drift **SD(t)**, varies directly with the “potential ballistic drag force”  $q(t)*S = \rho*V^2(t)*S/2$  in pounds. The ambient atmospheric density  $\rho$  varies with shooting conditions.

The rifle bullet's retained velocity **V(t)** depends upon its muzzle velocity **V(0)**, its mass **m**, and the integrated drag function **CD<sub>α</sub>** of that bullet. The bullet's cross-sectional area  $S = \pi*d^2/4$  varies with the square of the bullet's caliber **d**.

*Third*, the spin-drift **SD(t)** of the bullet is proportional to its yaw-of-repose angle **β<sub>R</sub>(t)** throughout its flight:

$$\beta_R(t) = (2\pi*g/t) \int [\omega_2(t)*V(t)]^{-1} dt$$

Both the coning rate **ω<sub>2</sub>(t)** and the forward velocity **V(t)** of the bullet always gradually decrease, continually increasing **β<sub>R</sub>(t)** throughout the bullet's flight. The coning rate **ω<sub>2</sub>(t)** is determined by the bullet's fixed inertial ratio **I<sub>y</sub>/I<sub>x</sub>** and by the remaining spin-rate **ω(t)** and slowly increasing gyroscopic stability **Sg** of the flying bullet.

The forward velocity **V(t)** of the flying bullet depends on its launch velocity **V(0)** and its coefficient of drag profile.

The yaw-of-repose attitude angle **β<sub>R</sub>(t)** is *increased* for bullets having larger numerical **I<sub>y</sub>/I<sub>x</sub>** ratios and higher initial stability **Sg**, but **β<sub>R</sub>(t)** is *decreased* by using faster twist-rate barrels and higher muzzle velocities **V(0)** to achieve that higher gyroscopic stability **Sg**.

*Fourth*, the spin-drift **SD(t)** is directly proportional to the small-yaw coefficient of lift **CL<sub>β</sub>(t)** of the bullet. Very-low-drag (VLD) and ultra-low-drag (ULD) bullet designs usually have correspondingly reduced coefficient-of-lift functions at all supersonic airspeeds.

*Fifth*, and lastly, the spin-drift **SD(t)** of the bullet is inversely proportional to the weight **Wt** (or mass **m**) of that bullet.

All else being equal, bullets made with lower average material densities, such as turned brass bullets, will weigh less and will suffer greater spin-drift.

These five **SD** effects combine multiplicatively in this analysis.

Some bullet and rifle design parameters recur in several of these different **SD** effects, and not always working in the same direction.

As modern long-range rifles and their bullets seem to be evolving toward lighter-weight, smaller-caliber, lower-drag bullets fired at higher muzzle velocities, these related incremental variations in design parameters combine algebraically to *reduce the spin-drift SD occurring on long-range targets*.

#### Disclaimers & Notices

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